Math 524: Linear Algebra

Notes #5 — Eigenvalues, Eigenvectors, and Invariant Subspaces

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Outline

Student Learning Targets, and Objectives

Target Invariant Subspaces

- **Objective** Know how the restriction operator and invariant subspaces are connected
- **Objective** Be familiar with the 1-D "line-type" subspaces and their connection with eigenvalues and eigenvectors.
- **Objective** Know, and be able to use, the fact that eigenvectors corresponding to distinct eigenvalues are linearly independent.

Target Eigenvalues, Eigenvectors, and Eigenspaces

- **Objective** Know that every operator on a finite-dimensional, nonzero, complex vector space has an eigenvalue and an upper-triangular matrix with respect to some basis
- **Objective** Know the the definitions of Eigenspaces of operators; and understand the discussion of how "collecting" enough eigenvalues can guarantee invertibility of an operator.

Introduction

We now turn our attention to one of the cornerstones of Linear Algebra, the study of **Operators on finite-dimensional vector spaces**.

Rewind (Operator, $\mathcal{L}(V)$)

- A linear map from a vector space to itself is called an **operator**.
- The notation $\mathcal{L}(V)$ denotes the set of all operators on V. In other words, $\mathcal{L}(V) = \mathcal{L}(V, V)$.

We will use our newly acquired abstract understanding of vector spaces and linear maps to the study of eigen-values and eigen-vectors.

Time-Target: 3×75-minute lectures.

Invariant Subspaces

Let $T \in \mathcal{L}(V)$. If we have [Divide-and-Conquer "Theorem"]

$$V = U_1 \oplus \cdots \oplus U_m$$

where each U_j is a proper subspace of V (*i.e.* $\dim(U_j) < \dim(V)$); then it is sufficient to understand the action of T on each U_j.

Notation (Restriction, $T|_{U_j}$)

$T|_{U_j}$ is the **restriction** of the linear map $T \in \mathcal{L}(V)$ to the subspace U_j.

This only makes sense if $T|_{U_j} : U_j \mapsto U_j$, or if you want $T|_{U_j} \in \mathcal{L}(U_j)$. Such subspaces get their own name...

Definition (Invariant Subspace)

Let $T \in \mathcal{L}(V)$. A subspace U of V is called **invariant** under T if $\forall u \in U \Rightarrow T(u) \in U$.

Invariant Subspaces

Example (Invariant Subspaces)

Suppose $T \in \mathcal{L}(V)$, then the following subspace are invariant under T

- $\{0\}$
  - If $u \in \{0\}$, then $u = 0$. [LINEARITY] $T(u) = T(0) = 0 \in \{0\}$.
- V
  - If $u \in V$, then — since $T \in \mathcal{L}(V)$ — $T(u) \in V$
- $\mathrm{null}(T)$ — (could be $\{0\}$)
  - If $u \in \mathrm{null}(T)$, then $T(u) = 0 \in \mathrm{null}(T)$ [LINEARITY]
- $\mathrm{range}(T)$ — (could be V)
  - If $u \in \mathrm{range}(T)$, then $T(u) \in \mathrm{range}(T)$, by definition of range.

Invariant Subspaces

Question

Must an operator $T \in \mathcal{L}(V)$ have any invariant subspaces other than $\{0\}$ and V?

— $\mathrm{null}(T)$ and $\mathrm{range}(T)$ do not necessarily provide useful insight.

We will see that the answer is yes, as long as $\dim(V) > 1$ (for $\mathbb{F} = \mathbb{C}$), or $\dim(V) > 2$ (for $\mathbb{F} = \mathbb{R}$).

Example

Let $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ is defined by $Dp = p'$, then for any fixed m, $\mathcal{P}_m(\mathbb{R})$ is an invariant subspace of $\mathcal{P}(\mathbb{R})$.

In this case:

$$\dim(\mathrm{null}(D)) = 1, \text{ and } \dim(\mathrm{range}(D)) = m = (\dim(\mathcal{P}_m(\mathbb{R})) - 1).$$

Eigenvalues and Eigenvectors :: Eigenvalues

We will look at invariant subspaces in careful detail; first we turn our attention to the case of invariant subspaces with $\dim = 1$.

Consider the 1-dimensional ("line"-type) subspaces: let $v \neq 0 \in V$, and define $U = \{\lambda v : \lambda \in \mathbb{F}\} \equiv \mathrm{span}(v)$.

If U is invariant under $T \in \mathcal{L}(V)$, then $T(v) \in U$ $(\forall v \in U)$, and hence $\exists \lambda \in \mathbb{F}$ such that

$$T(v) = \lambda v$$

The converse holds: if $T(v) = \lambda v$ for some $v \in V$, and $\lambda \in \mathbb{F}$, then $\mathrm{span}(v)$ is an invariant subspace of V under the linear map T.

Eigenvalues and Eigenvectors :: Eigenvalues

In the past, we have surely seen eigenvalues (and eigenvectors) defined for *matrices*, here we generalize the concept to operators on all finite-dimensional subspaces...

Definition (Eigenvalue)

Suppose $T \in \mathcal{L}(V)$. A scalar $\lambda \in \mathbb{F}$ is called an **eigenvalue** of T if there exists $v \neq 0 \in V$ such that $T(v) = \lambda v$.

Eigenvalues and Eigenvectors :: Eigenvalues

Theorem (Equivalent Conditions to be an Eigenvalue)

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{F}$. The following are equivalent:

(a) λ *is an eigenvalue of* T

(b) $T - \lambda I$ *is not injective*

(c) $T - \lambda I$ *is not surjective*

(d) $T - \lambda I$ *is not invertible*

Recall: $I \in \mathcal{L}(V) : I(v) = v, \forall v \in V$.

(a) $\Leftrightarrow$ (b), by rearranging $T(v) = \lambda v$

(b) $\Leftrightarrow$ (c) $\Leftrightarrow$ (d) by [Injectivity $\Leftrightarrow$ Surjectivity in Finite Dimensions (Notes#3.2)]

Eigenvalues and Eigenvectors :: Eigenvectors

Definition (Eigenvector)

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. A vector $v \in V$ is called an **eigenvector** of T corresponding to the eigenvalue λ if $v \neq 0$, and $T(v) = \lambda v$.

Note: Eigenvalues can be 0, but
Eigenvectors cannot be the zero-vector.
(!) Eigenvectors corresponding to $\lambda = 0$ come from $\mathrm{null}(T)$.

Since $T(v) = \lambda v$ if and only if $(T - \lambda I)v = 0$, $v \neq 0 \in V$ is an eigenvector of T corresponding to the eigenvalue λ if and only if $v \in \mathrm{null}(T - \lambda I)$.

Eigenvalues and Eigenvectors ::

Example (Rotation over $\mathbb{R}$ and $\mathbb{C}$)

Suppose $T \in \mathcal{L}(\mathbb{F}^2)$, is defined by $T(x,y) = (-y,x)$.

$\mathbb{F} = \mathbb{R}$ T is a counterclockwise rotation by $\pi/2$ about the origin in $\mathbb{R}^2$. There is no real scaling of a vector such that $(-y,x) = \lambda(x,y)$. T has no eigenvalue(s) and no eigenvector(s).

$\mathbb{F} = \mathbb{C}$ We are looking for $\lambda \in \mathbb{F}$ such that $(-y,x) = \lambda(x,y)$:

$$\left\{ \begin{array}{rcl} \lambda x &=& -y \\ \lambda y &=& x \end{array} \right. \quad \Rightarrow \quad -y = \lambda x = \lambda(\lambda y) = \lambda^2 y$$

EIGENVALUES: $\lambda^2 = -1 \Rightarrow \lambda = \pm i$.

EIGENVECTORS: $\left\{ \begin{array}{lcl} (-y,x) = +i(x,y) & \Rightarrow & (x,y) = (w,-wi) \\ (-y,x) = -i(x,y) & \Rightarrow & (x,y) = (w,+wi) \end{array} \right.$

Eigenvectors Corresponding to Distinct Eigenvalues are Linearly Independent

Theorem (Linearly Independent Eigenvectors)

Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \dots, \lambda_m$ are distinct eigenvalues of T, and $v_1, \dots, v_m$ are the corresponding eigenvectors; then $v_1, \dots, v_m$ is linearly independent.

Proof (Linearly Independent Eigenvectors)

[By Contradiction] Suppose $v_1, \dots, v_m$ is linearly dependent. Let k be the smallest positive integer such that $v_k \in \mathrm{span}(v_1, \dots, v_{k-1})$. We can find $a_1, \dots, a_{k-1} \in \mathbb{F}$ such that

$$\begin{array}{lrcl} (1) & v_k & = & a_1 v_1 + \cdots + a_{k-1} v_{k-1} \\ & T(v_k) & = & T(a_1 v_1 + \cdots + a_{k-1} v_{k-1}) \\ (2) & \lambda_k v_k & = & a_1 \lambda_1 v_1 + \cdots + a_{k-1} \lambda_{k-1} v_{k-1} \\ \hline \lambda_k(1) - (2) & 0 & = & a_1(\lambda_k - \lambda_1) v_1 + \cdots + a_{k-1}(\lambda_k - \lambda_{k-1}) v_{k-1} \end{array}$$

Since $v_1, \dots, v_{k-1}$ is linearly independent, λ_k is magically equal to all the distinct $\lambda_1, \dots, \lambda_{k-1}$. **Contradiction!**

Number of Eigenvalues $\leq \dim(V)$

Theorem (Number of Eigenvalues)

Suppose V is finite-dimensional. Then each operator on V has at most $\dim(V)$ distinct eigenvalues.

Proof (Number of Eigenvalues)

Let $m = \dim(V)$. We can find at most m linearly independent vectors in V; eigenvectors corresponding to distinct eigenvalues are linearly independent (by previous theorem); so we can find at most m eigenvectors; thus at most m distinct eigenvalues.

Restriction and Quotient Operators

If $T \in \mathcal{L}(V)$, and U is a subspace of V invariant under T, then U determines two other operators:

Definition (Restriction Operator $T|_U$; and Quotient Operator T/U)

Suppose $T \in \mathcal{L}(V)$, and U is a subspace of V invariant under T

- The **restriction operator** $T|_U \in \mathcal{L}(U)$ is defined by
$$T|_U(u) = T(u), \quad u \in U$$
- The **quotient operator** $T/U \in \mathcal{L}(V/U)$ is defined by
$$(T/U)(v + U) = T(v) + U, \quad v \in V$$

Restriction and Quotient Operators :: Example

Example

Define $T \in \mathcal{L}(\mathbb{F}^2)$ by $T(x,y) = (y,0)$. Let $U = \{(x,0) : x \in \mathbb{F}\}$

- U is invariant under T and $T|_U$ is the 0-operator on U:
  - $T(x,0) = (0,0) \in U$.

    So U is invariant under T and $T|_U$ is the 0-operator on U.
- $\nexists$ a subspace W of $\mathbb{F}^2$ that is invariant under T, and $U \oplus W = \mathbb{F}^2$.
  - Since $\dim(\mathbb{F}^2) = 2$, $\dim(U) = 1$, we must have $\dim(W) = 1$. If W is invariant under T, then all $w \in W$ are eigenvectors. However, the only eigenvalue is $\lambda = 0$, and U contains the corresponding eigenvectors.

    Thus W cannot be invariant under T.

Restriction and Quotient Operators :: Example 2/2

Example

Define $T \in \mathcal{L}(\mathbb{F}^2)$ by $T(x, y) = (y, 0)$. Let $U = \{(x, 0) : x \in \mathbb{F}\}$

- T/U is the 0-operator on $\mathbb{F}^2/U$:
  - $(x, y) \in \mathbb{F}^2$

$$\begin{aligned} (T/U)((x, y) + U) &= T(x, y) + U \\ &= (y, 0) + U \\ &= 0 + U \end{aligned}$$

the last equality holds because $(y, 0) \in U$.

This example shows that sometimes the restriction and quotient operators do not provide (enough) information about T. Here, both are the 0-operators on their respective spaces, even though T is not.

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e.g. 5A-{**12**}

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5A-12: Define $T \in \mathcal{L}(\mathcal{P}_4(\mathbb{R}))$ by $(Tp)(x) = xp'(x)\ \forall x \in \mathbb{R}$. Find all eigenvalue and eigenvectors of T.

⋇ Solution ⋇

We use the eigenvalue/eigenvector characterization $T(p) = \lambda p$ — here $xp'(x) = \lambda p(x)$. We can write any $p \in \mathcal{P}_4(\mathbb{R})$ in the form $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$, which gives us

$$xp'(x) = a_1 x + 2a_2 x^2 + 3a_3 x^3 + 4a_4 x^4 = \lambda(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4) = \lambda p(x)$$

In order for the equality to hold, the coeffcients for each power must be equal in the left and right expressions. Collecting those relations give us...

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$$\begin{cases} 0a_0 & = & \lambda a_0 \\ 1a_1 & = & \lambda a_1 \\ 2a_2 & = & \lambda a_2 \\ 3a_3 & = & \lambda a_3 \\ 4a_4 & = & \lambda a_4 \end{cases}$$

For any $j \in \{0,\ 1,\ 2,\ 3,\ 4\}$: a solution is given by

$$\{\ a_j \neq 0,\ \ \lambda = j,\ \ a_{k \neq j} = 0\ \},$$

which allows us to identify 5 eigenvalue-eigenvector pairs:

$$\{(0,1),\ (1,x),\ (2,x^2),\ (3,x^3),\ (4,x^4)\}$$

Technically, any non-zero scaling of the eigenvectors $\{1, x, x^2, x^3, x^4\}$ is also an eigenvector.

Polynomials Applied to Operators

The main reason that a richer theory exists for operators (which map a vector space into itself) than for more general linear maps is that operators can be raised to powers — composed with themselves / applied multiple times:

Definition (T^m)

Suppose $T \in \mathcal{L}(V)$ and m is a positive integer

- T^m is defined by $T^m = \underbrace{T \circ \cdots \circ T}_{m \text{ times}}$
- T^0 is defined to be the identity operator on V
- If T is invertible, with inverse T^{-1}, then $T^{-m} = (T^{-1})^m$

The Operator $p(T)$

Definition (The Operator $p(T)$)

Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$ is a polynomial given by

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m, \; z \in \mathbb{F}$$

Then $p(T)$ is the operator defined by

$$p(T) = a_0 I + a_1 T + a_2 T^2 + \cdots + a_m T^m$$

Example ("The Gateway to Differential Equations.")

Suppose $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ is the differentiation operator defined by $Dq = q'$, with p being the polynomial defined by $p(x) = x^2 + k$, then $p(D) = D^2 + k$, and

$$p(D)q = q'' + kq, \; \forall q \in \mathcal{P}(\mathbb{R})$$

$p(D)q = 0$ is the Helmholtz Equation (in 1D).

https://en.wikipedia.org/wiki/Helmholtz_equation

Product of Polynomials

Definition (Product of Polynomials)

If $p, q \in \mathcal{P}(\mathbb{F})$, then $pq \in \mathcal{P}(\mathbb{F})$ is the polynomial defined by

$$(pq)(z) = p(z)q(z), \;\; z \in \mathbb{F}$$

Theorem (Multiplicative Properties)

Suppose $p, q \in \mathcal{P}(\mathbb{F})$ *and* $T \in \mathcal{L}(V)$, *then*

- $(pq)(T) = p(T)q(T)$
- $p(T)q(T) = q(T)p(T)$

The proof is purely "mechanical" (distributive property + bookkeeping)

Existence of Eigenvalues

Theorem (Existence of Eigenvalues)

Every operator on a finite-dimensional, nonzero, complex vector space has an eigenvalue.

Proof (Existence of Eigenvalues)

Suppose V is a complex vector space with dimension $n > 0$ and $T \in \mathcal{L}(V)$. Let $v \neq 0 \in V$, then

$$v, T(v), T^2(v), \ldots, T^n(v)$$

is not linearly independent, because V has dimension n and we have $(n+1)$ vectors. Thus there exist complex numbers $a_0, a_1, \ldots, a_n$, such that

$$0 = a_0 v + a_1 T(v) + a_2 T^2(v) + \cdots + a_n T^n(v).$$

Not all $a_1, \ldots, a_n$ can be zero, since that would force $a_0 = 0$ (and this would make the $(n+1)$ vectors linearly independent)...

Existence of Eigenvalues

Proof (Existence of Eigenvalues)

Now, let the a's be the coefficients of a polynomial; which by the [FUNDAMENTAL THEOREM OF ALGEBRA] has a factorization

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n = c(z - \lambda_1) \cdots (z - \lambda_m)$$

where c is a nonzero complex number, $\lambda_j \in \mathbb{C}$, and the equation holds $\forall z \in \mathbb{C}$ (here m is not necessarily equal to n, because a_n may equal 0). We then have

$$\begin{aligned} 0 &= a_0 v + a_1 T(v) + a_2 T^2(v) + \cdots + a_n T^n(v) \\ &= (a_0 I + a_1 T + a_2 T^2 + \cdots + a_n T^n)(v) \\ &= c(T - \lambda_1 I) \cdots (T - \lambda_m I) v \end{aligned}$$

Thus $(T - \lambda_j I)$ is not injective for at least one j. $\Leftrightarrow$ T has an eigenvalue.

Upper-Triangular Matrices

Definition (Matrix of an Operator, $\mathcal{M}(T)$)

Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. The **matrix of T** with respect to this basis is the $(n \times n)$ matrix

$$\mathcal{M}(T) = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{bmatrix}$$

whose entries $a_{j,k}$ are defined by

$$T(v_k) = a_{1,k} v_1 + \cdots + a_{n,k} v_n$$

If the basis is not "obvious from context," then we use the notation $\mathcal{M}(T, (v_1, \ldots, v_n))$.

Upper-Triangular Matrices :: Comments

Note that matrices of operators are square, rather than the more general rectangular case which we considered earlier for linear maps.

If T is an operator on $\mathbb{F}^n$ and no basis is specified, assume that the basis in question is the standard basis. The jth column of $\mathcal{M}(T)$ is then T applied to the jth basis vector.

A central ~~goal~~ milestone of linear algebra is to show that given an operator $T \in \mathcal{L}(V)$, there exists a basis of V with respect to which T has a reasonably simple matrix.

For instance, we might try to choose a basis of V such that $\mathcal{M}(T)$ has many 0's.

Upper-Triangular Matrices

If V is a finite-dimensional complex vector space, there is a basis of V with respect to which the matrix of T looks like

	v	w_1	$\cdots$	w_{n-1}
v	λ	$*$	$*$	$*$
w_1	0	$*$	$*$	$*$
$\vdots$	$\vdots$	$*$	$*$	$*$
w_{n-1}	0	$*$	$*$	$*$

Let λ be an eigenvalue of T (existence is guaranteed); and let v be the corresponding eigenvector. Extend v to a basis of V: $v, w_1, \dots, w_{n-1}$ [Linearly Independent List Extends to a Basis (Notes#2)]. Then the matrix of T with respect to this basis has the form given.

Upper-Triangular Matrices

Definition (Diagonal of a Matrix)

The diagonal of a square matrix consists of the entries along the line from the upper left corner to the bottom right corner.

— The $a_{i,i}$-entries.

Definition (Upper-Triangular Matrix)

A matrix is called upper triangular if all the entries below the diagonal equal 0.

— $a_{i,j} = 0 \; \forall i > j$. "The strictly lower-triangular part is filled with zeros."

Conditions for Upper-Triangular Matrix

Theorem (Conditions for Upper-Triangular Matrix)

Suppose $T \in \mathcal{L}(V)$ and $v_1, \dots, v_n$ is a basis of V. Then the following are equivalent:

(a) *the matrix of T with respect to $v_1, \dots, v_n$ is upper triangular*

(b) $T(v_k) \in \mathrm{span}(v_1, \dots, v_k)$, $\forall k$

(c) $U_k = \mathrm{span}(v_1, \dots, v_k)$ *is invariant under* T $\forall k$

Proof (Conditions for Upper-Triangular Matrix)

(a) $\Leftrightarrow$ (b) follows from the definition, and (c) $\Rightarrow$ (b). The only part that requires work is (b) $\Rightarrow$ (c).

Suppose (b) holds. Fix $k \in \{1, \dots, n\}$ From (b) we know $T(v_i) \in \mathrm{span}\,(v_1, \dots, v_i) \subset \mathrm{span}\,(v_1, \dots, v_k)$, $i \in \{1, \dots, k\}$. Thus if $v = a_1 v_1 + \cdots + a_k v_k$, then $T(v) \in \mathrm{span}\,(v_1, \dots, v_k)$, which shows that $\mathrm{span}\,(v_1, \dots, v_k)$ is invariant under T.

Over $\mathbb{C}$, Every Operator has an Upper-Triangular Matrix

Theorem (Over $\mathbb{C}$, Every Operator has an Upper-Triangular Matrix)

Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some basis of V.

Comment

The result does **not** hold on real vector spaces, because the first vector in a basis with respect to which an operator has an upper-triangular matrix is an eigenvector of the operator. Thus if an operator on a real vector space has no eigenvalues, then there is no basis with respect to which the operator has an upper-triangular matrix.

We skip the proof... but fear not, Axler provides 2 proofs in the book (pp.149–150).

Determination of Invertibility from Upper-Triangular Matrix

The following two theorems indicate why we have gone through so much trouble to (isomorphically) link operators on abstract vector spaces to operators on $T \in \mathcal{L}(\mathbb{F}^n)$...

Theorem (Determination of Invertibility from Upper-Triangular Matrix)

Suppose $T \in \mathcal{L}(V)$ has an upper-triangular matrix with respect to some basis of V. Then T is invertible if and only if all the entries on the diagonal of that upper-triangular matrix are nonzero.

Theorem (Determination of Eigenvalues from Upper-Triangular Matrix)

Suppose $T \in \mathcal{L}(V)$ has an upper-triangular matrix with respect to some basis of V. Then the eigenvalues of T are precisely the entries on the diagonal of that upper-triangular matrix.

Unfortunately, identifying bases which reveal eigenvalues is non-trivial.

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e.g. 5B-{**4**, 5, 6, 10}

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5B-4: Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$.
Prove that $V = \mathrm{null}(P) \oplus \mathrm{range}(P)$.

⋇ (i) $\mathrm{null}(P) \cap \mathrm{range}(P) = \{0\}$ ⋇

Let $u \in \mathrm{null}(P) \cap \mathrm{range}(P)$. Then $P(u) = 0$, and $\exists w \in W : u = P(w)$.
Applying P to $u = P(w)$ gives

$$0 = \mathbf{P}(\mathbf{u}) = \mathbf{P}^2(\mathbf{w}) = P(w) = u,$$

hence the only vector in $\mathrm{null}(P) \cap \mathrm{range}(P)$ is $u = 0$.

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⋇ (ii) $V = \mathrm{null}(P) + \mathrm{range}(P)$ ⋇

Next, let $v \in V$, then

$$v = v + 0 = v + (P(v) - P(v)) = (v - P(v)) + P(v),$$

where

$$P(v-P(v)) = P(v)-P^2(v) = P(v)-P(v) = 0 \;\Rightarrow\; (v-P(v)) \in \mathrm{null}(P),$$

and, by definition

$$P(v) \in \mathrm{range}(P).$$

Thus

$$v = u + w, \quad u \in \mathrm{null}(P), \; w \in \mathrm{range}(P)$$

Since $v \in V$ was arbitrary, $V = \mathrm{null}(P) + \mathrm{range}(P)$.

⋇ (i) + (ii) $\Rightarrow V = \mathrm{null}(P) \oplus \mathrm{range}(P)$ ⋇

[Direct Sum of Two Subspaces (Notes#1)]

Eigenspaces and Diagonal Matrices

Definition (Diagonal Matrix)

A diagonal matrix is a square matrix that is 0 everywhere except possibly along the diagonal.

Note: zeros ARE allowed on the diagonal.

If an operator has a diagonal matrix with respect to some basis, then the entries along the diagonal are precisely the eigenvalues of the operator.

Definition (Eigenspace, $E(\lambda, T)$)

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. The **Eigenspace** of T corresponding to λ denoted $E(\lambda, T)$ is defined to be

$$E(\lambda, T) = \mathrm{null}(T - \lambda I)$$

$E(\lambda, T)$ is the set of all eigenvectors of T corresponding to λ, along with the 0 vector.

The Operator Restricted to an Eigenspace

The Operator Restricted to an Eigenspace:
If λ is an eigenvalue of $T \in \mathcal{L}(V)$, then

$$T|_{E(\lambda,T)}(v) = \lambda v, \ \ \forall v \in E(\lambda, T)$$

this indicates that eigenspaces are (non-trivial, and highly useful) **invariant subspaces**; and we get a very simple description (scalar multiplication) of the operator when restricted to such a subspace.

The is where our notation and language is starting to pay dividends!

Sum of Eigenspaces is a Direct Sum

Theorem (Sum of Eigenspaces is a Direct Sum)

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Suppose also that $\lambda_1, \dots, \lambda_m$ are distinct eigenvalues of T. Then

$$E(\lambda_1, T) + \cdots + E(\lambda_m, T)$$

is a direct sum:

$$E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$$

Furthermore,

$$\dim\left(E(\lambda_1, T)\right) + \cdots + \dim\left(E(\lambda_m, T)\right) \le \dim(V)$$

Eigenvalues and eigenspaces give us excellent understanding of operator behavior, so this is worth showing... However, we need a help-result...

Filling in a Gap... Dimension of a Direct Sum of Finite Dimensional Subspaces

Theorem (Dimension of a Direct Sum of Finite Dimensional Subspaces)

Suppose $U_1, \ldots, U_m$ are finite-dimensional subspaces of V such that $U_1 + \cdots + U_m$ is a direct sum. Then $U_1 \oplus \cdots \oplus U_m$ is finite-dimensional and

$$\dim(U_1 \oplus \cdots \oplus U_m) = \dim(U_1) + \cdots + \dim(U_m)$$

Proof (Dimension of a Direct Sum of Finite Dimensional Subspaces)

Let $W, U_1, \ldots, U_m$ be subspaces of V such that

$$W = U_1 \oplus \cdots \oplus U_m$$

Choose a (finite) basis $\{u_{j,k}\}$ for each U_j. Concatenate the bases into a single list $\{w_\ell\}$. By construction the (finite) list $\{w_\ell\}$ spans $W = U_1 + \cdots + U_m$. Thus W is finite dimensional... $\rightsquigarrow$

Filling in a Gap... Dimension of a Direct Sum of Finite Dimensional Subspaces

Proof (Dimension of a Direct Sum of Finite Dimensional Subspaces)

(!) We need to show that the vectors in $\{w_\ell\}$ are linearly independent (and thus a basis for W), so that the dimension of W equals the number of vectors in $\{w_\ell\}$, and therefore

$$\dim(W) = \dim(U_1) + \cdots + \dim(U_m).$$

Assume: $a_1 w_1 + \cdots + a_N w_N = 0$; we can group the sum in m groups (depending on what space U_j was the original source of the basis vector); and thus write this sum as $\widehat{u}_1 + \cdots + \widehat{u}_m = 0$, where each $\widehat{u}_j \in U_j$.

Since $W = U_1 \oplus \cdots \oplus U_m$, this forces $0 \equiv \widehat{u}_j \in U_j \ \forall j$; but since each such vector is formed by a linear combination of the basis vectors $\{u_{j,k}\}$, all the coefficients in each of those linear combinations must be 0; translated back to $a_1 w_1 + \cdots + a_N w_N = 0$, $a_i \equiv 0$, which makes $\{w_\ell\}$ linearly independent. $\surd$

Sum of Eigenspaces is a Direct Sum

With the help-result in hand, we can show the theorem:

Proof (Sum of Eigenspaces is a Direct Sum)

To show that $E(\lambda_1, T) + \cdots + E(\lambda_m, T)$ is a direct sum, let

$$u_1 + \cdots + u_m = 0$$

where $u_j \in E(\lambda_j, T)$. Since the eigenvectors corresponding to distinct eigenvalues are linearly independent [Linearly Independent Eigenvectors], we get $u_j \equiv 0$. Now, using [Condition for a Direct Sum (Notes#1)] this implies that $E(\lambda_1, T) + \cdots + E(\lambda_m, T)$ is a direct sum. Using the [help-result], we now get

$$\begin{aligned} \dim\left(E(\lambda_1, T) + \cdots + E(\lambda_m, T)\right) &= \\ \dim\left(E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)\right) &= \\ \dim\left(E(\lambda_1, T)\right) + \cdots + \dim\left(E(\lambda_m, T)\right) &\leq \dim(V) \end{aligned}$$

Diagonalizable Operators

Definition (Diagonalizable)

An operator $T \in \mathcal{L}(V)$ is called **diagonalizable** if the operator has a diagonal matrix with respect to some basis of V.

Theorem (Conditions Equivalent to Diagonalizability)

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\lambda_1, \dots, \lambda_m$ denote the distinct eigenvalues of T. Then the following are equivalent:

(a) *T is diagonalizable.*

(b) *V has a basis consisting of eigenvectors of T* — *"Eigenbasis"*

(c) *$\exists$ 1-D subspaces $U_1, \dots, U_n$ of V, each invariant under T, such that $V = U_1 \oplus \cdots \oplus U_n$*

(d) $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$

(e) $\dim(V) = \dim(E(\lambda_1, T)) + \cdots + \dim(E(\lambda_m, T))$

Conditions Equivalent to Diagonalizability

Proof (Conditions Equivalent to Diagonalizability)

(a)⇔(b) $T \in \mathcal{L}(V)$ has a diagonal matrix $\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ if and only if $T(v_k) = \lambda v_k$ for each k. $\surd$

(b)⇒(c) Suppose (b) holds; *i.e.* V has a basis $v_1, \ldots, v_n$ consisting of eigenvectors of T. For each k, let $U_k = \mathrm{span}\,(v_k)$. By construction each U_k is a 1-D subspace of V that is invariant under T. Because $v_1, \ldots, v_n$ is a basis of V, each vector in V can be written uniquely as a linear combination of $v_1, \ldots, v_n$. That is, each vector in V can be written uniquely as a sum $u_1 + \cdots + u_n$, where $u_k \in U_k$. Thus $V = U_1 \oplus \cdots \oplus U_n$. $\surd$

Conditions Equivalent to Diagonalizability

Proof (Conditions Equivalent to Diagonalizability)

(c)⇒(b) Suppose (c) holds; thus there are 1-dimensional subspaces $U_1, \dots, U_n$ of V, each invariant under T, such that $V = U_1 \oplus \cdots \oplus U_n$. $\forall k$, let $v_k \neq 0 \in U_k$. Then each v_k is an eigenvector of T. Because each vector in V can be written uniquely as a sum $u_1 + \cdots + u_n$, where $u_k = \alpha_k v_k \in U_k$, we see that $v_1, \dots, v_n$ is a basis of V. $\surd$

(b)⇒(d) Suppose (b) holds; thus V has a basis consisting of eigenvectors of T. Hence every vector in V is a linear combination of eigenvectors of T, which implies that

$$V = E(\lambda_1, T) + \cdots + E(\lambda_m, T)$$

Now [Sum of eigenspaces is a direct sum] shows that (d) holds:

$$V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T). \qquad \surd$$

Conditions Equivalent to Diagonalizability

Proof (Conditions Equivalent to Diagonalizability)

(d)⇒(e) [HELP-RESULT] $\surd$

(e)⇒(b) Suppose (e) holds, *i.e.* $\dim(V) = \dim(E(\lambda_1, T)) + \cdots + \dim(E(\lambda_m, T))$. Select a basis for each $E(\lambda_j, T)$; concatenate the basis into a list $v_1, \ldots, v_n$ of eigenvectors of T ($n = \dim(V)$, by (e)). For linear independence, suppose $a_1 v_1 + \cdots + a_n v_n = 0$; let u_j denote the sum of the group of vectors from $E(\lambda_j, T)$; and we get $u_1 + \cdots + u_m = 0$. Now, [LINEARLY INDEPENDENT EIGENVECTORS] forces $u_j = 0$, which in turn forces $a_i = 0$, which makes $v_1, \ldots, v_n$ linearly independent, and a basis for V by [LINEARLY INDEPENDENT LIST OF THE RIGHT LENGTH IS A BASIS (NOTES#2)]. $\surd$

We now have
(a) ⇔ (b) ⇔ (c)
(b) ⇒ (d) ⇒ (e) ⇒ (b)

which means (a) ⇔ (b) ⇔ (c) ⇔ (d) ⇔ (e) $\surd$

Diagonalizability is not Guaranteed

Unfortunately not every operator is diagonalizable. This can happen even on complex vector spaces, as was shown in one of our previous examples:

Rewind

The $T \in \mathcal{L}(\mathbb{F}^2)$ defined by $T(x,y) = (y,0)$ has a single eigenvalue $\lambda = 0$, and $E(\lambda = 0, T) = \{(x,0) : x \in \mathbb{F}\}$.
Since $\dim(\mathbb{F}^2) = 2$, and $\dim(E(0,T)) = 1$; we're out of luck

At some point (soon) we have to "do something" about non-diagonalizable operators.

Enough Eigenvalues Implies Diagonalizability

Theorem (Enough Eigenvalues Implies Diagonalizability)

If $T \in \mathcal{L}(V)$ has $n = \dim(V)$ distinct eigenvalues, then T is diagonalizable.

Proof (Enough Eigenvalues Implies Diagonalizability)

Let $T \in \mathcal{L}(V)$ have $n = \dim(V)$ distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. $\forall k$, let $v_k \in V$ be an eigenvector corresponding to λ_k. By [LINEARLY INDEPENDENT EIGENVECTORS] $v_1, \ldots, v_n$ is linearly independent, and by [LINEARLY INDEPENDENT LIST OF THE RIGHT LENGTH IS A BASIS (NOTES#2)] therefore a basis. With respect to this basis consisting of eigenvectors, T has a diagonal matrix.

Note: this is a one-way result $\Rightarrow$, not an if-and-only-if $\Leftrightarrow$.

$\langle\langle\langle$ Live Math $\rangle\rangle\rangle$

e.g. 5C-$\{\mathbf{8}\}$

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5C-8: Suppose $T \in \mathcal{L}(\mathbb{F}^5)$ and $\dim(E(8,T)) = 4$. Prove that $(T - 2I)$ or $(T - 6I)$ is invertible.

Solution

We remind ourselves that

- $E(\lambda, T) = \mathrm{null}(T - \lambda I)$ are the eigenspaces, and
- the sum of eigenspaces is a direct sum

Therefore

$$\underbrace{\dim(E(8,T))}_{4} + \underbrace{\dim(E(2,T))}_{\in Z^+} + \underbrace{\dim(E(6,T))}_{\in Z^+} \leq \dim(\mathbb{F}^5) = 5$$

which means that at least one (possibly both) of $\dim(E(2,T))$ and $\dim(E(6,T))$ must be zero. $\rightsquigarrow \exists\widehat{\lambda} \in \{2, 6\}$ so that $E(\widehat{\lambda}, T) = \{0\}$, making $\widehat{\lambda}$ not an eigenvalue, and $(T - \widehat{\lambda}I)$ invertible.

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Suggested Problems

5.A—1, 2, 3, 4, 8, 9, 10, 12

5.B—1(a), 4‡, 5*, 7, 10, 14, 15

5.C—1, 2, 8

‡ This problem has "something" to do with Orthogonal Projections. (Familiar on $\mathbb{R}^n$, but here expressed on general vector spaces); also a matrix for which $P^2 = P$ holds is called *idempotent*. This problem appears in slightly different form in [Math 543]

* This problem relates to the eigenvalue structure, and the reason we (in Math 254) look for diagonalizing similarity transformations

Assigned Homework — HW#5, Due Date in Canvas/Gradescope

5.A—4, 8, 10(a)

5.B—1(a), 7, 14, 15

5.C—1, 2

Note: Assignment problems are not official and subject to change until the first lecture on the chapter has been delivered (or virtually "scheduled.")

Upload homework to www.Gradescope.com

Eigenvectors Corresponding to Distinct Eigenvalues are Linearly Independent

Proof ((Alternative) Linearly Independent Eigenvectors)

Let $\in \mathcal{L}(V)$. Suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T, and $v_1, \ldots, v_m$ are the corresponding eigenvectors. Consider $c_1 v_1 + \cdots + c_m v_m = 0$, where $c_j \in \mathbb{F}$. Let k be the largest index so that $v_1, \ldots, v_k$ is linearly independent, but $v_1, \ldots, v_{k+1}$ is not and let (*) $c_1 v_1 + \cdots + c_k v_k = v_{k+1}$. Now:

$$\begin{aligned} \lambda_{k+1} v_{k+1} = T(v_{k+1}) &= T(c_1 v_1 + \cdots + c_k v_k) = c_1 T(v_1) + \cdots + c_k T(v_k) \\ &= c_1 \lambda_1 v_1 + \cdots + c_k \lambda_k v_k \quad (**) \end{aligned}$$

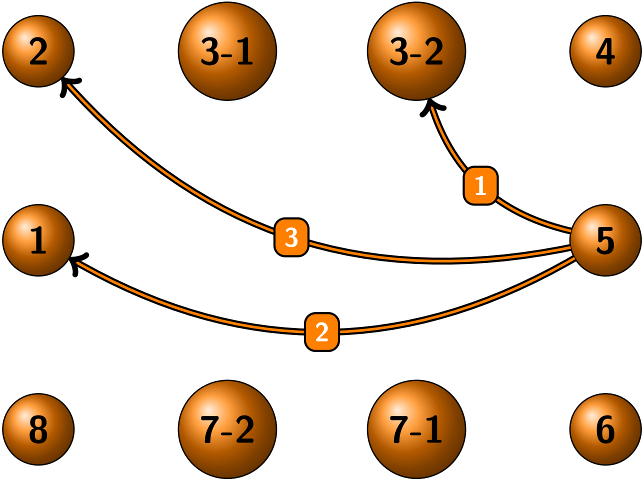
Multiplying (*) by λ_{k+1} and subtract from (**):

$$c_1 \underbrace{(\lambda_1 - \lambda_{k+1})}_{\neq 0} v_1 + \cdots + c_k \underbrace{(\lambda_k - \lambda_{k+1})}_{\neq 0} v_k = 0$$

Since $v_1, \ldots, v_k$ are linearly independent, $c_1 = \cdots = c_k = 0$, which makes $v_{k+1} = 0$, but eigenvectors cannot be the zero vector; hence there is no such value k, and the vectors are linearly independent.

(It's the same proof, dressed up in a Halloween costume!)

Explicit References to Previous Theorems or Definitions (with count)



Explicit References to Previous Theorems or Definitions