Math 524: Linear Algebra

Notes #7.1 — Operators on Inner Product Spaces

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Outline

What We Know So Far — Operators, $T \in \mathcal{L}(V)$

- Some operators are diagonalizable, *i.e.* $\exists\, \mathfrak{B}(V)$ so that $\mathcal{M}(T, \mathfrak{B}(V))$ is diagonal

$$\Rightarrow\; V \;=\; \bigoplus_{k=1}^{\dim(V)} U_k \;=\; \bigoplus_{k=1}^{m} E(\lambda_k, T), \quad m \leq \dim(V)$$

- It is always possible to find an orthonormal basis $\mathfrak{B}(V) = b_1, \ldots, b_n$ so that $\mathcal{M}(T, \mathfrak{B}(V))$ is upper triangular
  $\Rightarrow$ $W_k = \mathrm{span}(b_1, \ldots, b_k)$ are nested subspaces: $W_{k-1} \subset W_k$, $\dim(W_k) = k$

- We now, in the next 8 lectures, seek to build better understanding in the huge void between *"some are diagonalizable"* and *"all are upper triangularizable."*

Student Learning Targets, and Objectives

Target Adjoint [Operator] — T^*

- **Objective** Be able to state the definition of the adjoint [operator]; and manipulate inner product expressions to obtain the adjoint T^* given an operator T
- **Objective** Know the definition of, and useful properties of Self-Adjoint, and Normal, Operators

Target Real and Complex Spectral Theorems

- **Objective** Know under what circumstances operators over the $\mathbb{R}$eal and $\mathbb{C}$omplex fields have orthonormal eigenbases; with respect to which the matrix of the operator is diagonal.

Time-Target: $2\times$75-minute lectures.

Introduction

We now look at operators $T \in \mathcal{L}(V)$ on inner product spaces, $V = \{\text{vector space, with } \langle v, w \rangle : V \times V \mapsto \mathbb{F}\}$.

$\Uparrow$

Inner Products $\Rightarrow$ Norms, Orthogonality, Gram-Schmidt, RRT

$\Uparrow$

Linear Maps
Operators $\Rightarrow$ Eigen-Values/Vectors; Invariant Subspaces

$\Uparrow$

Finite Dimensional Vector Spaces

Adjoints

Definition (Adjoint, T^*)

Suppose $T \in \mathcal{L}(V, W)$. The **adjoint** of T is the function $T^* : W \mapsto V$, such that $\forall v \in V$, and $\forall w \in W$:

$$\langle Tv,\, w \rangle = \langle v,\, T^* w \rangle.$$

$T^* w$ is uniquely defined due to [Riesz Representation Theorem (Notes#6)]

Note that in the case when $V \neq W$, the inner product $\langle Tv,\, w \rangle$ is on the space W, and $\langle v,\, T^* w \rangle$ is on the space V.

We will shortly show that $T^* \in \mathcal{L}(W, V)$, *i.e.* it is a linear map.

There is another "adjoint" in linear algebra... we will not speak about it, shhhhhh!!!

Example#1: Find the Adjoint

Example (Find T^* —)

Let $T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$ be defined by

$$T(x_1, x_2, x_3) = (x_2 + 3x_3, 2x_1).$$

Solution: $T^* : \mathbb{R}^2 \mapsto \mathbb{R}^3$, fix a point $(y_1, y_2) \in \mathbb{R}^2$; then $\forall (x_1, x_2, x_3) \in \mathbb{R}^3$:

$$\begin{aligned}
\langle (x_1, x_2, x_3),\, T^*(y_1, y_2) \rangle &= \langle T(x_1, x_2, x_3),\, (y_1, y_2) \rangle \\
&= \langle (x_2 + 3x_3, 2x_1),\, (y_1, y_2) \rangle \\
&= x_2 y_1 + 3x_3 y_1 + 2x_1 y_2 \\
&= \langle (x_1, x_2, x_3),\, (2y_2, y_1, 3y_1) \rangle \\
\hline
T^*(y_1, y_2) &= (2y_2, y_1, 3y_1)
\end{aligned}$$

Example#2: Find the Adjoint

Example (Find T^* —)

Fix $u \in V$, and $x \in W$. Let $T \in \mathcal{L}(V, W)$ be defined by

$$Tv = \langle v,\, u \rangle x$$

Solution: Fix $w \in W$. Then $\forall v \in V$, we have

$$\begin{aligned}
\langle v,\, T^* w \rangle &= \langle Tv,\, w \rangle \\
&= \langle \langle v,\, u \rangle x,\, w \rangle \\
&= \langle v,\, u \rangle \langle x,\, w \rangle \\
&= \langle v,\, \langle x,\, w \rangle^* u \rangle \\
&= \langle v,\, \langle w,\, x \rangle u \rangle \\
\hline
T^* w &= \langle w,\, x \rangle u
\end{aligned}$$

In both examples, T^* turned out to be a linear map.
This is true in general:

The Adjoint is a Linear Map

Theorem (The Adjoint is a Linear Map)

If $T \in \mathcal{L}(V,W)$, then $T^* \in \mathcal{L}(W,V)$.

Proof (The Adjoint is a Linear Map)

Suppose $T \in \mathcal{L}(V,W)$. Fix $w_1, w_2 \in W$, if $v \in V$, then

$$\begin{aligned}\langle v,\, T^*(w_1+w_2)\rangle &= \langle Tv,\, w_1+w_2\rangle &&= \langle Tv,\, w_1\rangle + \langle Tv,\, w_2\rangle \\ &= \langle v,\, T^*w_1\rangle + \langle v,\, T^*w_2\rangle &&= \langle v,\, T^*w_1 + T^*w_2\rangle\end{aligned}$$

$$T^*(w_1+w_2) = T^*w_1 + T^*w_2 \qquad \text{[Additivity (Notes\#3.1)]}$$

Next, fix $w \in W$, and $\lambda \in \mathbb{F}$, if $v \in V$, then

$$\begin{aligned}\langle v,\, T^*(\lambda w)\rangle &= \langle Tv,\, \lambda w\rangle &&= \lambda^*\langle Tv,\, w\rangle \\ &= \lambda^*\langle v,\, T^*w\rangle &&= \langle v,\, \lambda T^*w\rangle\end{aligned}$$

$$T^*(\lambda w) = \lambda T^* w \qquad \text{[Homogeneity (Notes\#3.1)]}$$

Thus, T^* is a linear map.

Properties of the Adjoint

Properties (Properties of the Adjoint)

- $(S+T)^* = S^* + T^*\ \forall S, T \in \mathcal{L}(V,W)$;
- $(\lambda T)^* = \lambda^* T^*\ \forall \lambda \in \mathbb{F}$, and $\forall T \in \mathcal{L}(V,W)$;
- $(T^*)^* = T\ \forall T \in \mathcal{L}(V,W)$;
- $I^* = I$, where I is the identity operator on V;
- $(ST)^* = T^*S^*\ \forall T \in \mathcal{L}(V,W), \forall S \in \mathcal{L}(W,U)$.

The proofs are standard plug-into-the-definitions-and-move-things-around; and are left as an "exercise."

These properties probably look vaguely familiar?

(Think about matrices..... and transposes?)

Null Space and Range of the Adjoint

Theorem (Null Space and Range of T^*)

Suppose $T \in \mathcal{L}(V,W)$, then:

$$\begin{aligned}(a)&\quad \text{null}(T^*) &= (\text{range}(T))^\perp \\ (b)&\quad \text{range}(T^*) &= (\text{null}(T))^\perp \\ (c)&\quad \text{null}(T) &= (\text{range}(T^*))^\perp \\ (d)&\quad \text{range}(T) &= (\text{null}(T^*))^\perp\end{aligned}$$

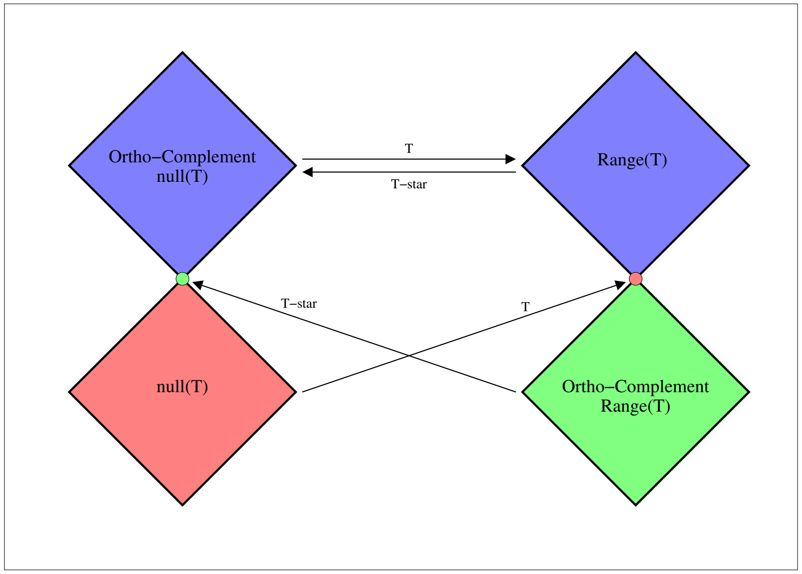
Proof (Null Space and Range of T^*)

(a) Let $w \in W$, then:

$$\begin{aligned}w \in \text{null}(T^*) &\iff T^*(w) = 0 \\ &\iff \langle v,\, T^*(w)\rangle = 0,\ \forall v \in V \\ &\iff \langle T(v),\, w\rangle = 0,\ \forall v \in V \\ &\iff w \in (\text{range}(T))^\perp\end{aligned}$$

(b), (c), (d) similar...

Null Space and Range of the Adjoint, Visualized



Conjugate Transpose, Hermitian Transpose

Definition (Conjugate Transpose)

The **conjugate transpose** of an $(m \times n)$ matrix is the $(n \times m)$ matrix obtained by interchanging the rows and columns and then taking the complex conjugate of each entry; *i.e.* $a_{ij} \mapsto a_{ji}^*$.

Notation (Conjugate Transpose)

For $A \in \mathbb{F}^{m\times n}$, we let $A^* \in \mathbb{F}^{n\times m}$ be the conjugate transpose of A.

Sometimes you see the notation A^H to indicate the Hermitian (Conjugate) transpose.

When $\mathbb{F} = \mathbb{R}$, the conjugate transpose is just the transpose.

The Matrix of T^*

Theorem (The Matrix of T^*)

Let $T \in \mathcal{L}(V, W)$, and let $v_1, \dots, v_n$ be an orthonormal basis of V, and $w_1, \dots, w_m$ be an orthonormal basis of W. Then

$$\mathcal{M}(T^*, (w_1, \dots, w_m), (v_1, \dots, v_n)) = \mathcal{M}(T, (v_1, \dots, v_n), (w_1, \dots, w_m))^*$$

In the above, it is **absolutely essential** for the bases of V and W to be orthonormal.

The adjoint of a linear map itself does not depend on the choice of basis; but the matrices of a linear map and its adjoint depend strongly on the choice of bases. *This is one of the compelling reasons why we develop our linear algebra toolbox in a more abstract rather than matrix-centered way.*

The Matrix of T^*

Proof (The Matrix of T^*)

The entries of $\mathcal{M}(T, (v_1, \dots, v_n), (w_1, \dots, w_m))$ are the coefficients of

$$T(v_k) = \langle T(v_k),\, w_1 \rangle w_1 + \cdots + \langle T(v_k),\, w_m \rangle w_m$$

i.e. $\mathcal{M}(T)_{ij} = \langle T(v_i),\, w_j \rangle$.

Likewise, the entries of $\mathcal{M}(T^*, (w_1, \dots, w_m), (v_1, \dots, v_n))$ are the coefficients of

$$T^*(w_k) = \langle T^*(w_k),\, v_1 \rangle v_1 + \cdots + \langle T^*(w_k),\, v_n \rangle v_n.$$

$\mathcal{M}(T^*)_{ij} = \langle T^*(w_i),\, v_j \rangle = \langle w_i,\, T(v_j) \rangle = (\langle T(v_j),\, w_i \rangle)^* = (\mathcal{M}(T)_{ji})^*$

Self-Adjoint Operators

We now consider operators $T \in \mathcal{L}(V)$, on inner product spaces (*i.e.* vector spaces with an inner product).

Definition (Self-Adjoint (Hermitian))

An operator $T \in \mathcal{L}(V)$ is called **self-adjoint** if $T = T^*$, *i.e.* $T \in \mathcal{L}(V)$ is **self-adjoint** if and only if

$$\langle T(v),\, w \rangle = \langle v,\, T(w) \rangle$$

$\forall v, w \in V$.

Sources of Hermitian / Self-Adjoint Operators

Comment

Physicists are sometimes(?) a bit careless with mathematical language, but in particular the field of quantum mechanics is full of Hermitian / Self-Adjoint operators — usually on infinite-dimensional Hilbert spaces.

Comment

Roughly speaking, the study of "linear algebra" on infinite-dimensional spaces is branded "Functional Analysis."

—

Functional Analysis is the meeting point of linear algebra and analysis, with a good measure$^{\text{funny?}}$ of topology sprinkled in.

Eigenvalues of Self-Adjoint Operators are Real

Theorem (Eigenvalues of Self-Adjoint Operators are Real)

Every eigenvalue of a self-adjoint operator is real.

Proof (Eigenvalues of Self-Adjoint Operators are Real)

Suppose $T \in \mathcal{L}(V)$ is self-adjoint. Let λ be an eigenvalue of T, and let v be an eigenvector: $T(v) = \lambda v$. Then

$$\lambda\|v\|^2 = \langle \lambda v,\, v\rangle = \langle T(v),\, v\rangle = \langle v,\, T(v)\rangle = \langle v,\, \lambda v\rangle = \lambda^*\|v\|^2$$

Since $\lambda = \lambda^*$, $\lambda \in \mathbb{R}$.

Note that if we are restricting ourselves to $\mathbb{F} = \mathbb{R}$ then the theorem is true by definition (restriction), so it is of interest (use) only in the case $\mathbb{F} = \mathbb{C}$.

Over $\mathbb{C}$, $T(v) \perp v\ \forall v \in V$ Only for the 0-Operator

Theorem (Over $\mathbb{C}$, $T(v) \perp v\ \forall v \in V$ Only for the 0-Operator)

Suppose V is a complex inner products space, and $T \in \mathcal{L}(V)$. Then if $\langle T(v),\, v\rangle = 0\ \forall v \in V$, then $T = 0$.

Note that the theorem is not true for real inner products spaces: consider the rotation by $\pi/2$ in $\mathbb{R}^2$.

Over $\mathbb{C}$, $T(v) \perp v\ \forall v \in V$ Only for the 0-Operator

Proof (Over $\mathbb{C}$, $T(v) \perp v\ \forall v \in V$ Only for the 0-Operator)

We need to show $\langle T(u),\, w\rangle = 0$, $\forall u, w \in V$. We rewrite this inner product in an appropriately complicated way:

$$\begin{aligned}\langle T(u),\, w\rangle &= \frac{\langle T(u+w),\, u+w\rangle - \langle T(u-w),\, u-w\rangle}{4} \\ &\quad + i\frac{\langle T(u+iw),\, u+iw\rangle - \langle T(u-iw),\, u-iw\rangle}{4}\end{aligned}$$

each term on the right-hand-side is of the form $\langle T(v),\, v\rangle$, so if $\langle T(v),\, v\rangle = 0\ \forall v \in V$, then it follows that $\langle T(u),\, w\rangle = 0$, $\forall u, w \in V$, and thus $T = 0$ (let $w = T(u)$).

For peace of mind, let's just verify the equality!

Post-Proof: Verifying the Equality

$$
\begin{array}{llcl}
+1 & \langle T(u+w),\, u+w\rangle & = & \langle T(u),\, u\rangle + \langle T(u),\, w\rangle + \langle T(w),\, u\rangle + \langle T(w),\, w\rangle \\
-1 & \langle T(u-w),\, u-w\rangle & = & \langle T(u),\, u\rangle - \langle T(u),\, w\rangle - \langle T(w),\, u\rangle + \langle T(w),\, w\rangle \\
\hline
 & & = & 2\langle T(u),\, w\rangle + 2\langle T(w),\, u\rangle \\
 & & & \\
+i & \langle T(u+iw),\, u+iw\rangle & = & \langle T(u),\, u\rangle + \langle T(u),\, iw\rangle + \langle T(iw),\, u\rangle + \langle T(iw),\, iw\rangle \\
 & & = & \langle T(u),\, u\rangle - i\langle T(u),\, w\rangle + i\langle T(w),\, u\rangle + \langle T(w),\, w\rangle \\
-i & \langle T(u-iw),\, u-iw\rangle & = & \langle T(u),\, u\rangle + \langle T(u),\, -iw\rangle + \langle T(-iw),\, u\rangle + \langle T(-iw),\, -iw\rangle \\
 & & = & \langle T(u),\, u\rangle + i\langle T(u),\, w\rangle - i\langle T(w),\, u\rangle + \langle T(w),\, w\rangle \\
\hline
 & & = & 2\langle T(u),\, w\rangle - 2\langle T(w),\, u\rangle \\
 & & & \\
\hline\hline
 & & = & 4\langle T(u),\, w\rangle
\end{array}
$$

Self-Adjoint and Normal Operators
The Spectral Theorem
Adjoints
Self-Adjoint Operators
Normal Operators

Over $\mathbb{C}$, $\langle T(v),\, v\rangle \in \mathbb{R}\ \forall v \in V$ Only for Self-Adjoint Operators

"Self-adjoint operators behave like real numbers...":

Theorem (Over $\mathbb{C}$, $\langle T(v),\, v\rangle \in \mathbb{R}\ \forall v \in V$ Only for Self-Adjoint Operators)

Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$. Then T is self-adjoint if and only if

$$\langle T(v),\, v\rangle \in \mathbb{R}$$

$\forall v \in V$.

Self-Adjoint and Normal Operators
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Over $\mathbb{C}$, $\langle T(v),\, v\rangle \in \mathbb{R}\ \forall v \in V$ Only for Self-Adjoint Operators

Proof (Over $\mathbb{C}$, $\langle T(v),\, v\rangle \in \mathbb{R}\ \forall v \in V$ Only for Self-Adjoint Operators)

Let $v \in V$, then

$$
\begin{aligned}
\langle T(v),\, v\rangle - \langle T(v),\, v\rangle^* &= \langle T(v),\, v\rangle - \langle v,\, T(v)\rangle \\
&= \langle T(v),\, v\rangle - \langle T^*(v),\, v\rangle \\
&= \langle (T - T^*)(v),\, v\rangle
\end{aligned}
$$

$\Leftarrow$ If $\langle T(v),\, v\rangle \in \mathbb{R}\ \forall v \in V$, then the left-hand-side is 0; so $\langle (T - T^*)(v),\, v\rangle = 0 \Rightarrow T = T^*$
[OVER $\mathbb{C}$, $T(v) \perp v\ \forall v \in V$ ONLY FOR THE 0-OPERATOR]

$\Rightarrow$ If $T = T^*$, then the right-hand-side is 0. Thus $\langle T(v),\, v\rangle = \langle T(v),\, v\rangle^* \Rightarrow \langle T(v),\, v\rangle \in \mathbb{R}$.

Self-Adjoint and Normal Operators
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If $T = T^*$ and $\langle T(v),\, v\rangle = 0$, $\forall v \in V$, then $T = 0$.

On a real inner product space V, a nonzero operator T might satisfy $\langle T(v),\, v\rangle = 0$, $\forall v \in V$. However, this cannot happen for a self-adjoint operator:

Theorem (If $T = T^*$ and $\langle T(v),\, v\rangle = 0$, $\forall v \in V$, then $T = 0$)

Suppose T is a self-adjoint operator on V such that $\langle T(v),\, v\rangle = 0$, $\forall v \in V$, then $T = 0$.

[OVER $\mathbb{C}$, $T(v) \perp v\ \forall v \in V$ ONLY FOR THE 0-OPERATOR] covered the case for complex inner product spaces without the self-adjointness property; so we only have to cover the real product spaces with the self-adjointness property:

If $T = T^*$ and $\langle T(v), v\rangle = 0$, $\forall v \in V$, then $T = 0$

Proof (If $T = T^*$ and $\langle T(v), v\rangle = 0$, $\forall v \in V$, then $T = 0$)

If $u, w \in V$, then

$$\langle T(u), w\rangle = \frac{\langle T(u+w), u+w\rangle - \langle T(u-w), u-w\rangle}{4}$$

the equality holds due to self-adjointness and the fact that we are in a real inner product space, see top of [SLIDE 21], and use:

$$\langle T(w), u\rangle \overset{sa}{=} \langle w, T(u)\rangle \overset{\mathbb{R}}{=} \langle T(u), w\rangle$$

again, each term on the right-hand-side is of the form $\langle T(v), v\rangle$; hence $\langle T(v), v\rangle = 0\ \forall v \in V \Rightarrow \langle T(u), w\rangle = 0\ \forall u, w \in V \Rightarrow T = 0$.

Normal Operators

Definition (Normal Operator)

- An operator on an inner product space is called **normal** if it commutes with its adjoint.
- $T \in \mathcal{L}(V)$ is normal if
$$TT^* = T^*T$$

Every self-adjoint operator is normal ($T^*T = T^2 = TT^*$), but the converse does not hold:

Example (Non Self-Adjoint, but Normal Operator)

Let $T \in \mathcal{L}(\mathbb{F}^2)$ be the operator with matrix (wrt standard basis)

$$\mathcal{M}(T) = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$$

Since $3 \neq (-3)^*$ the operator is not self adjoint, but

$$\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix}$$

shows that T^*T and TT^* have the same matrix $\Rightarrow T^*T = TT^* \Rightarrow T$ is normal.

Properties of Normal Operators

Theorem (T is Normal if-and-only-if $\|T(v)\| = \|T^*(v)\|\ \forall v \in V$)

An operator $T \in \mathcal{L}(V)$ is normal if and only if

$$\|T(v)\| = \|T^*(v)\|, \forall v \in V$$

Proof (T is Normal if-and-only-if $\|T(v)\| = \|T^*(v)\|\ \forall v \in V$)

Let $T \in \mathcal{L}(V)$, then

$$\begin{aligned}
T \text{ is normal} \quad &\Leftrightarrow \quad T^*T - TT^* = 0 \\
&\Leftrightarrow \quad \langle (T^*T - TT^*)(v), v\rangle = 0 && \forall v \in V \\
&\Leftrightarrow \quad \langle (T^*T(v), v\rangle = \langle (TT^*(v), v\rangle && \forall v \in V \\
&\Leftrightarrow \quad \langle (T(v), T(v)\rangle = \langle (T^*(v), T^*(v)\rangle && \forall v \in V \\
&\Leftrightarrow \quad \|T(v)\|^2 = \|T^*(v)\|^2 && \forall v \in V
\end{aligned}$$

Properties of Normal Operators :: Eigenvectors of T and T^*

Theorem (For T Normal, T and T^* Have the Same Eigenvectors)

Suppose $T \in \mathcal{L}(V)$ is normal; $v \in V$ is an eigenvector of T with eigenvalue λ. Then v is also an eigenvector of T^* with eigenvalue λ^*.

Proof (For T Normal, T and T^* Have the Same Eigenvectors)

Since $T \in \mathcal{L}(V)$ is normal, so is $T - \lambda I$; using the previous theorem we have

$$0 = \|(T - \lambda I)(v)\| = \|(T - \lambda I)^*(v)\| = \|(T^* - \lambda^* I)(v)\|$$

hence $T^*v = \lambda^*v$.

Properties of Normal Operators :: Orthogonal Eigenvectors

Don't forget: Every self-adjoint operator is normal.

Theorem (Orthogonal Eigenvectors for Normal Operators)

Suppose $T \in \mathcal{L}(V)$ is normal. Then eigenvectors of T corresponding to distinct eigenvalues are orthogonal.

[This is a fairly big deal. Consider the impact on invariant subspaces, etc...]
[Orthogonality is the "ultimate" linear independece! Also the path to computational efficiency.]

Proof (Orthogonal Eigenvectors for Normal Operators)

Let $(\lambda_1, v_1), (\lambda_2, v_2)$ be distinct eigen-value/vector pairs of T, then:

$$\begin{aligned}(\lambda_1 - \lambda_2)\langle v_1,\, v_2\rangle &= \langle \lambda_1 v_1,\, v_2\rangle - \langle v_1,\, \lambda_2^* v_2\rangle \\ &= \langle T(v_1),\, v_2\rangle - \langle v_1,\, T^*(v_2)\rangle \\ &= 0 \quad \text{[Using the definition of } T^*]\end{aligned}$$

This shows $\langle v_1,\, v_2\rangle = 0$, *i.e.* $v_1 \perp v_2$.

⟨⟨⟨ Live Math ⟩⟩⟩

e.g. 7A-{2, 3, 5, 7, 12, **14**}

Live Math :: Covid-19 Version 7A-14

7A-14: Suppose T is a normal operator on V. Suppose also that $v, w \in V$ satisfy the equations

$$\|v\| = \|w\| = 2, \quad T(v) = 3v, \quad T(w) = 4w.$$

Show that $\|T(v+w)\| = 10$.

The given information shows that $(3, v)$ and $(4, w)$ are two eigen-value/vector pairs. [Orthogonal Eigenvectors for Normal Operators] says that $v \perp w$ (hinting at the use of the [Pythagorean Theorem])...

Putting it all together gives:

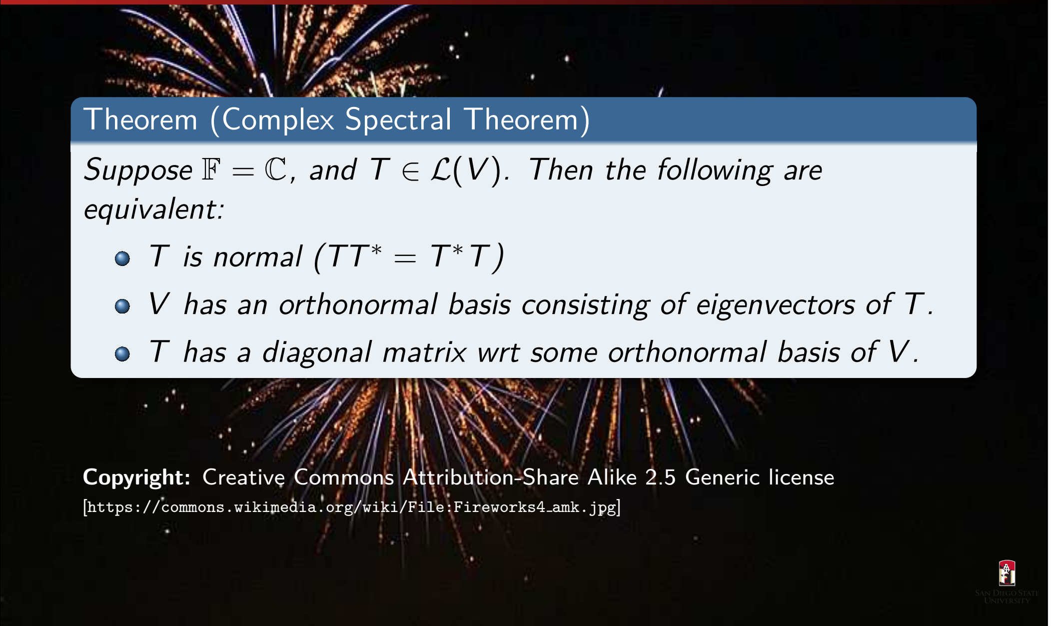
$$\begin{aligned}\|T(v+w)\| &= \|3v + 4w\| = \sqrt{9\|v\|^2 + 16\|w\|^2} \\ &= \sqrt{9 \cdot 4 + 16 \cdot 4} = \sqrt{100} \\ &= 10\end{aligned}$$

The Spectral Theorem

Rewind (+ Comments)

- A diagonal matrix is a square matrix that is 0 everywhere except possibly along the diagonal.
- An operator on V has a diagonal matrix wrt a basis if and only if the basis consists of eigenvectors of the operator [Conditions Equivalent to Diagonalizability (Notes#5)]
- The most easily understood operators on V are those for which there is an orthonormal basis of V wrt which the operator has a diagonal matrix. These are the operators $T \in \mathcal{L}(V)$ such that there is an orthonormal basis of V consisting of eigenvectors of T.
- Next, we look at the Spectral Theorem, which characterizes these operators as the normal operators when $\mathbb{F} = \mathbb{C}$, and as the self-adjoint operators when $\mathbb{F} = \mathbb{R}$.

Complex Spectral Theorem



Theorem (Complex Spectral Theorem)

Suppose $\mathbb{F} = \mathbb{C}$, and $T \in \mathcal{L}(V)$. Then the following are equivalent:

- *T is normal ($TT^* = T^*T$)*
- *V has an orthonormal basis consisting of eigenvectors of T.*
- *T has a diagonal matrix wrt some orthonormal basis of V.*

Complex Spectral Theorem

Example ($T \in \mathcal{L}(\mathbb{F}^2)$ — Normal, but not Self-adjoint)

We again consider $T \in \mathcal{L}(\mathbb{F}^2)$ with matrix (wrt standard basis)

$$\mathcal{M}(T) = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$$

An orthonormal basis of $\mathbb{C}^2$ consisting of eigenvectors of $\mathcal{M}(T)$ is given by $\mathfrak{B}(\mathbb{F}^2) = \left\{ \frac{1}{\sqrt{2}}(i,1), \frac{1}{\sqrt{2}}(-i,1) \right\}$, and

$$\mathcal{M}(T; \mathfrak{B}(\mathbb{F}^2)) = \begin{bmatrix} 2+3i & 0 \\ 0 & 2-3i \end{bmatrix}$$

Complex Spectral Theorem

Proof (Complex Spectral Theorem)

(c)⇒(a): Suppose T has a diagonal matrix wrt some orthonormal basis, $\mathfrak{B}(V)$ of V, *i.e.* $\mathcal{M}(T; \mathfrak{B}(V))$ is diagonal. $\mathcal{M}(T^*; \mathfrak{B}(V)) = \mathcal{M}(T; \mathfrak{B}(V))^*$ is also diagonal. Any two diagonal matrices commute, thus $TT^* = T^*T$.

(a)⇒(c): Suppose $TT^* = T^*T$. [Schur's Theorem (Notes#6)] guarantees ∃ an orthogonal basis $v_1, \dots, v_n$ of V so that

$$\mathcal{M}(T; (v_1, \dots, v_n)) = \begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ & \ddots & \vdots \\ 0 & & a_{n,n} \end{bmatrix}$$

Now, $\|T(v_1)\|^2 = \|T^*(v_1)\|^2$ since $TT^* = T^*T$, but

$$\begin{aligned} \|T(v_1)\|^2 &= |a_{1,1}|^2 \\ \|T^*(v_1)\|^2 &= |a_{1,1}|^2 + \underbrace{|a_{1,2}|^2 + \cdots + |a_{1,n}|^2}_{\Rightarrow \text{All must be zero}} \end{aligned}$$

Next, $\|T(v_2)\|^2 = \|T^*(v_2)\|^2$ shows in the same way that $|a_{2,3}| = \cdots = |a_{2,n}| = 0$; and in the same way, all non-diagonal elements are zero; and therefore $\mathcal{M}(T; (v_1, \dots, v_n))$ is diagonal.
Also, $T(v_i) = a_{i,i}v_i$, so the basis vector are eigenvectors ⇔ (b).

The Real Spectral Theorem

Rewind (Complete the Square)

Let $b, c \in \mathbb{R} : b^2 < 4c$, then

$$x^2 + bx + c = \left(x + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right) > 0.$$

In particular $(x^2 + bx + c)^{-1}$ is well-defined, or "$(x^2 + bx + c)$ is an invertible real number."

Now, we replace x with a self-adjoint operator...

Invertible Quadratic (Operator) Expressions

Theorem (Invertible Quadratic (Operator) Expressions)

Suppose $T \in \mathcal{L}(V)$ is self-adjoint, and $b, c \in \mathbb{R} : b^2 < 4c$, then

$$T^2 + bT + cI$$

is invertible.

Proof (Invertible Quadratic (Operator) Expressions)

Let $v \in V \setminus \{0\}$, then

$$\begin{aligned}
\langle (T^2 + bT + cI)(v), v \rangle &= \langle T^2(v), v \rangle + b\langle T(v), v \rangle + c\langle v, v \rangle \\
&= \langle T(v), T(v) \rangle + b\langle T(v), v \rangle + c\|v\|^2 \\
&\geq \|T(v)\|^2 - |b|\, \|T(v)\|\, \|v\| + c\|v\|^2 \qquad \text{[CBS (\#6)]} \\
&= \left(\|T(v)\| - \frac{|b|\, \|v\|}{2} \right)^2 + \left(c - \frac{b^2}{4} \right) \|v\|^2 \\
&> 0.
\end{aligned}$$

Thus $(T^2 + bT + cI)(v) \neq 0 \Rightarrow (T^2 + bT + cI)$ is injective $\Rightarrow$ invertible.

[For $\mathcal{L}(V)$: Injectivity $\Leftrightarrow$ Surjectivity in Finite Dimensions (Notes#3.2)]

Self-Adjoint Operators Have Eigenvalues

Rewind ([Existence of Eigenvalues (Notes#5)])

(!) Every operator on a finite-dimensional, nonzero, complex vector space has an eigenvalue. (!)

Theorem (Self-Adjoint Operators Have Eigenvalues)

Suppose $V \neq \{0\}$, and $T \in \mathcal{L}(V)$ is a self-adjoint operator. Then T has an eigenvalue.

This theorem tells us something new for $\mathbb{R}$eal inner product spaces.

Self-Adjoint Operators Have Eigenvalues

Proof (Self-Adjoint Operators Have Eigenvalues)

Let V be a real inner product space; $n = \dim(V)$, and $v \in V \setminus \{0\}$, then the list of $(n+1)$ vectors

$$v, T(v), T^2(v), \ldots, T^n(v)$$

cannot be linearly independent. $\Rightarrow \exists a_0, \ldots, a_n$ (not all zero):

$$0 = a_0 v + a_1 T(v) + a_2 T^2(v) + \cdots + a_n T^n(v),$$

let $p(x) = \sum_{k=0}^{n} a_k x^k \in \mathcal{P}_n(\mathbb{R})$ — which makes the above statement $p(T)v = 0$. Now, $p(x)$ [Factorization of a Polynomial Over $\mathbb{R}$ (Notes#4)] can be factored:

$$p(x) = c \prod_{k=1}^{M} (x^2 + b_k x + c_k) \prod_{\ell=1}^{m} (x - \lambda_\ell)$$

where $c \neq 0$; $b_i, c_i, \lambda_i \in \mathbb{R}$; $b_i^2 < 4c_i$; $M + m \geq 1$; $\forall x \in \mathbb{R}$.

Proof (Self-Adjoint Operators Have Eigenvalues)

Proof (Self-Adjoint Operators Have Eigenvalues)

Now, [Invertible Quadratic (Operator) Expressions] guarantees that

$$(T^2 + a_k T + c_k I), \; k = 1, \ldots, M$$

are invertible operators, and ($m > 0$ since $p(x) = 0$ has at least one solution)

$$0 = \left(\prod_{\ell=1}^{m} (T - \lambda_\ell I) \right) (v)$$

Thus $(T - \lambda_\ell I)$ is not injective for at least one $\ell \in \{1, \ldots, m \geq 1\}$; which means that T has an eigenvalue.

Self-Adjoint Operators and Invariant Subspaces

Theorem (Self-Adjoint Operators and Invariant Subspaces)

Suppose $T \in \mathcal{L}(V)$ is self-adjoint and U is a subspace of V that is invariant under T. Then

(a) *$U^\perp$ is invariant under T;*

(b) *$T|_U \in \mathcal{L}(U)$ is self-adjoint*

(c) *$T|_{U^\perp} \in \mathcal{L}(U^\perp)$ is self-adjoint*

Proof (Self-Adjoint Operators and Invariant Subspaces)

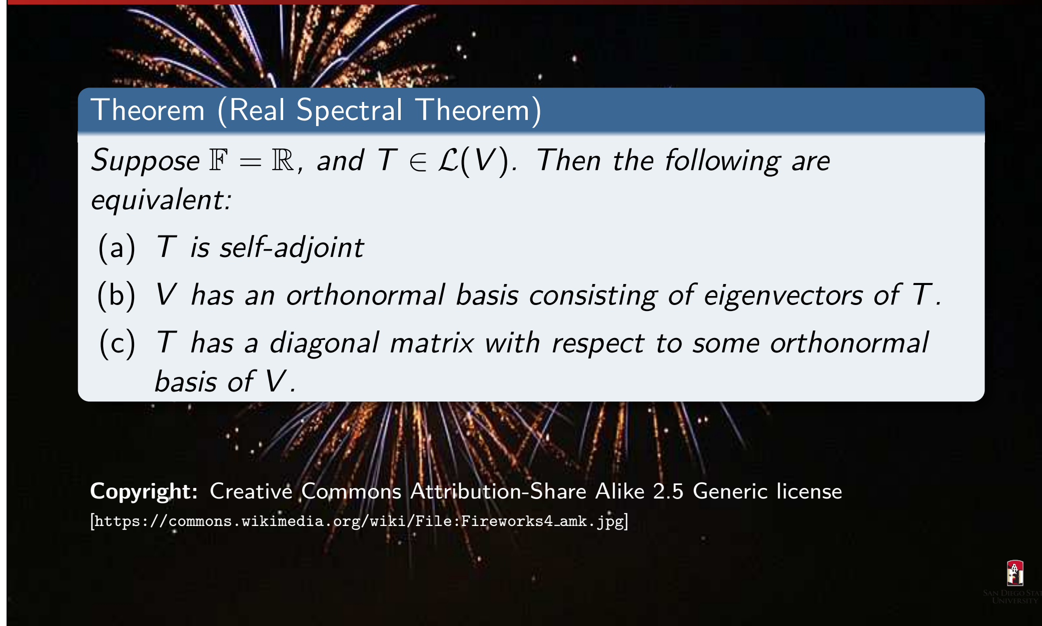
(a) Let $v \in U^\perp, u \in U$, then

$$\langle T(v),\, u\rangle \overset{sa}{=} \langle v,\, T(u)\rangle \overset{T(u)\in U}{=} 0 \Rightarrow T(v) \in U^\perp$$

(b) If $u, v \in U$, then $\langle T|_U(u),\, v\rangle = \langle T(u),\, v\rangle = \langle u,\, T(v)\rangle = \langle u,\, T|_U(v)\rangle$

(c) If $u, v \in U^\perp$, then $\langle T|_{U^\perp}(u),\, v\rangle = \langle T(u),\, v\rangle = \langle u,\, T(v)\rangle = \langle u,\, T|_{U^\perp}(v)\rangle$

Real Spectral Theorem



Theorem (Real Spectral Theorem)

Suppose $\mathbb{F} = \mathbb{R}$, and $T \in \mathcal{L}(V)$. Then the following are equivalent:

(a) *T is self-adjoint*

(b) *V has an orthonormal basis consisting of eigenvectors of T.*

(c) *T has a diagonal matrix with respect to some orthonormal basis of V.*

Real Spectral Theorem

Proof (Real Spectral Theorem)

(c $\Rightarrow$ a) Suppose (c) : T has a diagonal matrix with respect to some orthonormal basis of V. A real diagonal matrix equals its transpose. Therefore $T = T^*$, and thus T is self-adjoint. $\Rightarrow$ (a).

(a $\Rightarrow$ b) If $\dim(V) = 1$, then (a $\Rightarrow$ b); when $\dim(V) > 1$, and (INDUCTIVE HYPOTHESIS) that (a $\Rightarrow$ b) for all real product spaces $W : \dim(W) < \dim(V)$ — Let (a) $T \in \mathcal{L}(V)$ be self-adjoint, and let u be an eigenvector of T with $\|u\| = 1$. [SELF-ADJOINT OPERATORS HAVE EIGENVALUES]

Then $U = \operatorname{span}(u)$ is a 1-D subspace of V, invariant under T; $T|_{U^\perp}$ is self-adjoint [SELF-ADJOINT OPERATORS AND INVARIANT SUBSPACES]; $\dim(U^\perp) = \dim(V) - 1 < \dim(V)$; therefore $\exists$ an orthonormal basis of $U^\perp$ consisting of eigenvectors of $T|_{U^\perp}$. Adding u to this basis given an orthonormal basis of V consisting of eigenvectors of T. $\Rightarrow$ (b)

(b $\Rightarrow$ c) $\mathcal{M}(T)$ with respect to an orthonormal eigen-basis of V is a diagonal matrix. *(That's the point of finding an eigen-basis!)*

Real and Complex Spectral Theorems

Rewind (Complex Spectral Theorem)

Suppose $\mathbb{F} = \mathbb{C}$, and $T \in \mathcal{L}(V)$. Then the following are equivalent:

- T is normal ($TT^* = T^*T$)
- V has an orthonormal basis consisting of eigenvectors of T.
- T has a diagonal matrix wrt some orthonormal basis of V.

Rewind (Real Spectral Theorem)

Suppose $\mathbb{F} = \mathbb{R}$, and $T \in \mathcal{L}(V)$. Then the following are equivalent:

- T is self-adjoint ($T = T^*$)
- V has an orthonormal basis consisting of eigenvectors of T.
- T has a diagonal matrix wrt some orthonormal basis of V.

Real and Complex Spectral Theorems

Comment (Complex Spectral Theorem)

If $\mathbb{F} = \mathbb{C}$, then the Complex Spectral Theorem gives a complete description of the normal operators on V.

A complete description of the self-adjoint operators on V then easily follows — *they are the normal operators on V whose eigenvalues all are real.*

Comment (Real Spectral Theorem)

If $\mathbb{F} = \mathbb{R}$, then the Real Spectral Theorem gives a complete description of the self-adjoint operators on V.

A a complete description of the normal operators on V are forthcoming.

[Normal Operators and Invariant Subspaces (Notes#7.1–Preview)]

"Preview"

Preview (Normal Operators and Invariant Subspaces)

Suppose V is an inner product space, $T \in \mathcal{L}(V)$ is normal, and U is a subspace of V invariant under T. Then

(a) $U^\perp$ is invariant under T;
(b) U is invariant under T^*;
(c) $(T|_U)^* = (T^*)|_U$;
(d) $T|_U \in \mathcal{L}(U)$, and $T|_{U^\perp} \in \mathcal{L}(U^\perp)$ are normal operators.

⟨⟨⟨ Live Math ⟩⟩⟩

e.g. 7B-{3, **7**, 9, 15}

Live Math :: Covid-19 Version 7B-7

7B-7: Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$ is a normal operator such that $T^9 = T^8$. Prove that T is self-adjoint, and $T^2 = T$.

Note: There's nothing magical about 8, and 9...

By [CST], there exists an orthonormal basis $u_1, \dots, u_n$ of V such that $T(u_k) = \lambda_k u_k$ (where λ_k, $k = 1, \dots, n$ are the eigenvalues).

Applying T repeatedly on both sides of the eigen-relation gives $T^8(u_k) = \lambda_k^8 u_k$, and $T^9(u_k) = \lambda_k^9 u_k$; which by the given property $T^9 = T^8$, means $\lambda_k^8 = \lambda_k^9$. The only possibilities are $\lambda_k \in \{0, 1\} \in \mathbb{R}$.

Since the eigenvalues are real, T is self-adjoint. Also,

$$T^2(u_k) = \underbrace{\lambda_k^2 u_k \quad = \quad \lambda_k u_k}_{\text{since } \lambda_k \in \{0,1\}} = T(u_k)$$

Hence, $T^2 = T$.

Suggested Problems

7.A — **1**, 2, 3, **4**, 5, **6**, 7, 12, **14**

7.B — **2**, 3, **6**, 7, 9, 15

Assigned Homework — HW#7.1, Due Date in Canvas/Gradescope

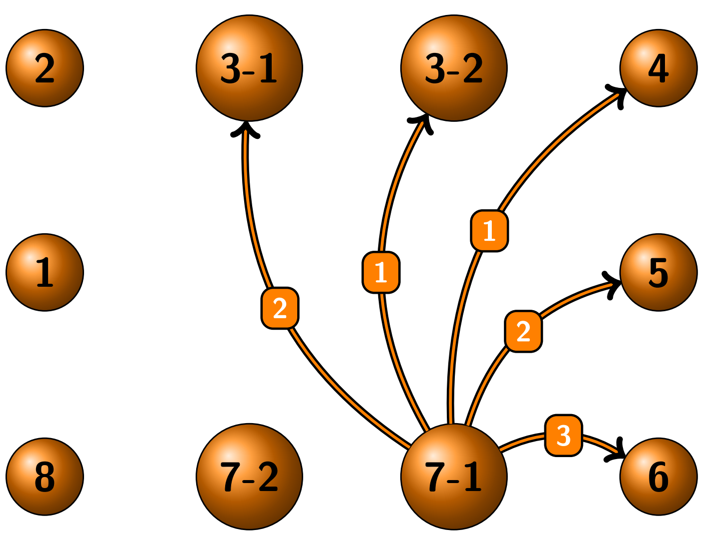
7.A—1, 4, 6, 14

7.B—2, 6

Note: Assignment problems are not official and subject to change until the first lecture on the chapter has been delivered (or virtually "scheduled.")

Upload homework to www.Gradescope.com

Explicit References to Previous Theorems or Definitions (with count)



Explicit References to Previous Theorems or Definitions