

# Numerical Analysis and Computing

Lecture Notes #10  
— Approximation Theory —  
Discrete Least Squares Approximation

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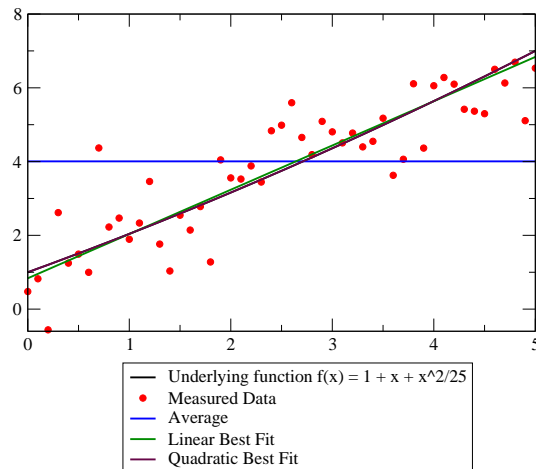
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## Introduction: Matching a Few Parameters to a Lot of Data.

Sometimes we get a lot of data, **many observations**, and want to fit it to a simple model.



PDF-link: [code](#).

## Why a Low Dimensional Model?

Low dimensional models (e.g. low degree polynomials) are **easy to work with**, and are quite **well behaved** (high degree polynomials can be quite oscillatory.)

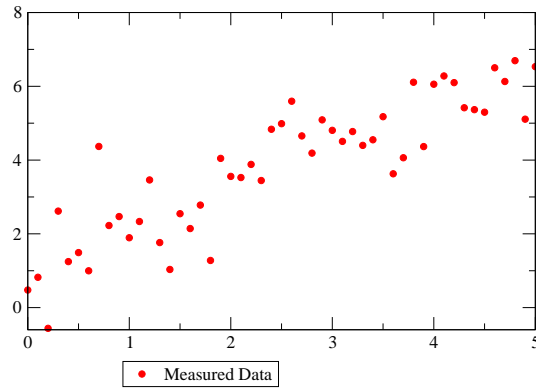
All measurements are **noisy**, to some degree. Often, we want to use a large number of measurements in order to “average out” random noise.

**Approximation Theory** looks at two problems:

- [1] Given a data set, find the best fit for a model (i.e. in a class of functions, find the one that best represents the data.)
- [2] Find a simpler model approximating a given function.

## Interpolation: A Bad Idea?

We can probably agree that trying to interpolate this data set:



with a 50th degree polynomial is not the best idea in the world...  
Even fitting a cubic spline to this data gives wild oscillations!  
**[I tried, and it was not pretty!]**

## What is the Size of the Residual?

There are many possible choices, e.g.

- The abs-sum of the deviations:

$$E_1 = \sum_{i=0}^n |r_i| \Leftrightarrow E_1 = \|\tilde{\mathbf{r}}\|_1$$

- The sum-of-the-squares of the deviations:

$$E_2 = \sqrt{\sum_{i=0}^n |r_i|^2} \Leftrightarrow E_2 = \|\tilde{\mathbf{r}}\|_2$$

- The largest of the deviations:

$$E_\infty = \max_{0 \leq i \leq n} |r_i| \Leftrightarrow E_\infty = \|\tilde{\mathbf{r}}\|_\infty$$

In most cases, the **sum-of-the-squares** version is the easiest to work with. (From now on we will focus on this choice...)

## Defining “Best Fit” — the Residual.

We are going to **relax** the requirement that the approximating function must pass through all the data points.

**Now we need a measurement of how well our approximation fits the data.** — A definition of “best fit.”

If  $f(x_i)$  are the measured function values, and  $a(x_i)$  are the values of our approximating functions, we can define a function,  $r(x_i) = f(x_i) - a(x_i)$  which measures the deviation (**residual**) at  $x_i$ . Notice that  $\tilde{\mathbf{r}} = \{r(x_0), r(x_1), \dots, r(x_n)\}^T$  is a vector.

**Notation:** From now on,  $f_i = f(x_i)$ ,  $a_i = a(x_i)$ , and  $r_i = r(x_i)$ .  
Further,  $\tilde{\mathbf{f}} = \{f_0, f_1, \dots, f_n\}^T$ ,  $\tilde{\mathbf{a}} = \{a_0, a_1, \dots, a_n\}^T$ ,  
and  $\tilde{\mathbf{r}} = \{r_0, r_1, \dots, r_n\}^T$ .

## Discrete Least Squares Approximation

We have chosen the sum-of-squares measurement for errors. Lets find the **constant** that best fits the data, *minimize*

$$E(C) = \sum_{i=0}^n (f_i - C)^2.$$

If  $C^*$  is a minimizer, then  $E'(C^*) = 0$  [**derivative at a max/min is zero**]

$$E'(C) = -\sum_{i=0}^n 2(f_i - C) = -2 \underbrace{\sum_{i=0}^n f_i + 2(n+1)C}_{\text{Set =0, and solve for C}}, \quad E''(C) = \underbrace{2(n+1)}_{\text{Positive}}$$

hence

$$C^* = \frac{1}{n+1} \sum_{i=0}^n f_i, \quad \text{it is a min since } E''(C^*) = 2(n+1) > 0.$$

is the constant that best the fits the data. (**Note:**  $C^*$  is the average.)

## Discrete Least Squares: Linear Approximation.

The form of Least Squares you are most likely to see: **Find the Linear Function**,  $p_1(x) = a_0 + a_1x$ , **that best fits the data**. The error  $E(a_0, a_1)$  we need to minimize is:

$$E(a_0, a_1) = \sum_{i=0}^n [(a_0 + a_1x_i) - f_i]^2.$$

The first partial derivatives with respect to  $a_0$  and  $a_1$  better be zero at the minimum:

$$\begin{aligned} \frac{\partial}{\partial a_0} E(a_0, a_1) &= 2 \sum_{i=0}^n [(a_0 + a_1x_i) - f_i] = 0 \\ \frac{\partial}{\partial a_1} E(a_0, a_1) &= 2 \sum_{i=0}^n x_i [(a_0 + a_1x_i) - f_i] = 0. \end{aligned}$$

We “massage” these expressions to get the **Normal Equations...**

## Quadratic Model, $p_2(x)$

For the quadratic polynomial  $p_2(x) = a_0 + a_1x + a_2x^2$ , the error is given by

$$E(a_0, a_1, a_2) = \sum_{i=0}^n [a_0 + a_1x_i + a_2x_i^2 - f_i]^2$$

At the minimum (best model) we must have

$$\begin{aligned} \frac{\partial}{\partial a_0} E(a_0, a_1, a_2) &= 2 \sum_{i=0}^n [(a_0 + a_1x_i + a_2x_i^2) - f_i] = 0 \\ \frac{\partial}{\partial a_1} E(a_0, a_1, a_2) &= 2 \sum_{i=0}^n x_i [(a_0 + a_1x_i + a_2x_i^2) - f_i] = 0 \\ \frac{\partial}{\partial a_2} E(a_0, a_1, a_2) &= 2 \sum_{i=0}^n x_i^2 [(a_0 + a_1x_i + a_2x_i^2) - f_i] = 0. \end{aligned}$$

## Linear Approximation: The Normal Equations

$p_1(x)$

$$\begin{cases} a_0(n+1) + a_1 \sum_{i=0}^n x_i = \sum_{i=0}^n f_i \\ a_0 \sum_{i=0}^n x_i + a_1 \sum_{i=0}^n x_i^2 = \sum_{i=0}^n x_i f_i. \end{cases}$$

Since **everything except  $a_0$  and  $a_1$**  is known, this is a 2-by-2 system of equations.

$$\begin{bmatrix} (n+1) & \sum_{i=0}^n x_i \\ \sum_{i=0}^n x_i & \sum_{i=0}^n x_i^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^n f_i \\ \sum_{i=0}^n x_i f_i \end{bmatrix}.$$

## Quadratic Model: The Normal Equations

$p_2(x)$

Similarly for the quadratic polynomial  $p_2(x) = a_0 + a_1x + a_2x^2$ , the normal equations are:

$$\begin{cases} a_0(n+1) + a_1 \sum_{i=0}^n x_i + a_2 \sum_{i=0}^n x_i^2 = \sum_{i=0}^n f_i \\ a_0 \sum_{i=0}^n x_i + a_1 \sum_{i=0}^n x_i^2 + a_2 \sum_{i=0}^n x_i^3 = \sum_{i=0}^n x_i f_i. \\ a_0 \sum_{i=0}^n x_i^2 + a_1 \sum_{i=0}^n x_i^3 + a_2 \sum_{i=0}^n x_i^4 = \sum_{i=0}^n x_i^2 f_i. \end{cases}$$

**Note:** Even though the model is quadratic, the resulting (normal) equations are **linear**. — The model is linear in its parameters,  $a_0$ ,  $a_1$ , and  $a_2$ .

## The Normal Equations — As Matrix Equations.

We rewrite the Normal Equations as:

$$\begin{bmatrix} (n+1) & \sum_{i=0}^n x_i & \sum_{i=0}^n x_i^2 \\ \sum_{i=0}^n x_i & \sum_{i=0}^n x_i^2 & \sum_{i=0}^n x_i^3 \\ \sum_{i=0}^n x_i^2 & \sum_{i=0}^n x_i^3 & \sum_{i=0}^n x_i^4 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^n f_i \\ \sum_{i=0}^n x_i f_i \\ \sum_{i=0}^n x_i^2 f_i \end{bmatrix}.$$

It is not immediately obvious, but this expression can be written in the form  $\mathbf{A}^T \mathbf{A} \tilde{\mathbf{a}} = \mathbf{A}^T \tilde{\mathbf{f}}$ . Where the matrix  $A$  is very easy to write in terms of  $x_j$ . [\[Jump Forward\]](#).

## Building a Solvable System from $\mathbf{A} \tilde{\mathbf{a}} = \tilde{\mathbf{f}}$

We cannot immediately solve the linear system

$$\mathbf{A} \tilde{\mathbf{a}} = \tilde{\mathbf{f}}$$

when  $A$  is a rectangular matrix  $(n+1)$ -by- $(m+1)$ ,  $m \neq n$ .

We can generate a solvable system by multiplying both the left- and right-hand-side by  $A^T$ , i.e.

$$\mathbf{A}^T \mathbf{A} \tilde{\mathbf{a}} = \mathbf{A}^T \tilde{\mathbf{f}}$$

Now, the matrix  $A^T A$  is a square  $(m+1)$ -by- $(m+1)$  matrix, and  $A^T \tilde{\mathbf{f}}$  an  $(m+1)$ -by-1 vector.

Let's take a closer look at  $A^T A$ , and  $A^T \tilde{\mathbf{f}}$ ...

## The Polynomial Equations in Matrix Form

$p_m(x)$

We can express the  $m$ th degree polynomial,  $p_m(x)$ , evaluated at the points  $x_i$ :

$$a_0 + a_1 x_i + a_2 x_i^2 + \dots + a_m x_i^m = f_i, \quad i = 0, \dots, n$$

as a product of an  $(n+1)$ -by- $(m+1)$  matrix,  $A$  and the  $(m+1)$ -by-1 vector  $\tilde{\mathbf{a}}$  and the result is the  $(n+1)$ -by-1 vector  $\tilde{\mathbf{f}}$ , usually  $n \gg m$ :

$$\underbrace{\begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^m \\ 1 & x_1 & x_1^2 & \dots & x_1^m \\ 1 & x_2 & x_2^2 & \dots & x_2^m \\ 1 & x_3 & x_3^2 & \dots & x_3^m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^m \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_m \end{bmatrix}}_{\tilde{\mathbf{a}}} = \underbrace{\begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{bmatrix}}_{\tilde{\mathbf{f}}}.$$

## Computing $A^T A$ .

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ x_0 & x_1 & x_2 & x_3 & \dots & x_n \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 & \dots & x_n^2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ x_0^m & x_1^m & x_2^m & x_3^m & \dots & x_n^m \end{bmatrix} \begin{bmatrix} 1 & x_0 & x_0^2 & \dots & x_0^m \\ 1 & x_1 & x_1^2 & \dots & x_1^m \\ 1 & x_2 & x_2^2 & \dots & x_2^m \\ 1 & x_3 & x_3^2 & \dots & x_3^m \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^m \end{bmatrix} = \begin{bmatrix} n+1 & \sum_{i=0}^n x_i^1 & \dots & \sum_{i=0}^n x_i^m \\ \sum_{i=0}^n x_i^1 & \sum_{i=0}^n x_i^2 & \dots & \sum_{i=0}^n x_i^{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{i=0}^n x_i^m & \sum_{i=0}^n x_i^{m+1} & \dots & \sum_{i=0}^n x_i^{2m} \end{bmatrix}.$$

## Computing $A^T f$ .

$$\begin{bmatrix} 1 & 1 & 1 & 1 & \dots & 1 \\ x_0 & x_1 & x_2 & x_3 & \dots & x_n \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 & \dots & x_n^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_0^m & x_1^m & x_2^m & x_3^m & \dots & x_n^m \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^n f_i \\ \sum_{i=0}^n x_i f_i \\ \sum_{i=0}^n x_i^2 f_i \\ \vdots \\ \sum_{i=0}^n x_i^m f_i \end{bmatrix}$$

We have recovered the Normal Equations...

[\[Jump Back\]](#).

## Discrete Least Squares: Matlab Example.

I used this code to generate the data for the plots on [slide 2](#).

```
x = (0:0.1:5)';           % The x-vector
f = 1+x+x.^2/25;         % The underlying function
n = randn(size(x));      % Random perturbations
fn = f+n;                % Add randomness
A = [x ones(size(x))];  % Build A for linear fit
%a = (A'*A)\(A'*fn);    % Solve: Using Normal Eqns.
a = A\fn;                % Solve: Better, Equivalent
p1 = polyval(a,x);       % Evaluate
A = [x.^2 x ones(size(x))]; % A for quadratic fit
%a = (A'*A)\(A'*fn);    % Solve: Using Normal Eqns.
a = A\fn;                % Solve: Better, Equivalent
p2 = polyval(a,x);       % Evaluate
```

## Discrete Least Squares: A Simple, Powerful Method.

Given the data set  $(\tilde{\mathbf{x}}, \tilde{\mathbf{f}})$ , where  $\tilde{\mathbf{x}} = \{x_0, x_1, \dots, x_n\}^T$  and  $\tilde{\mathbf{f}} = \{f_0, f_1, \dots, f_n\}^T$ , we can quickly find the best polynomial fit for **any** specified polynomial degree!

**Notation:** Let  $\tilde{\mathbf{x}}^j$  be the vector  $\{x_0^j, x_1^j, \dots, x_n^j\}^T$ .

**E.g.** to compute the best fitting polynomial of degree 3,  $p_3(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ , define:

$$A = \begin{bmatrix} | & | & | & | \\ \tilde{\mathbf{1}} & \tilde{\mathbf{x}} & \tilde{\mathbf{x}}^2 & \tilde{\mathbf{x}}^3 \\ | & | & | & | \end{bmatrix}, \quad \text{and compute } \underbrace{\tilde{\mathbf{a}} = (A^T A)^{-1} (A^T \tilde{\mathbf{f}})}_{\substack{\text{not necessarily} \\ \text{like this}}}$$

## But... I do not want to fit a polynomial!!!

Fitting an **exponential model**  $g(x) = be^{cx}$  to the given data  $\tilde{\mathbf{d}}$ , is quite straight-forward.

First, re-cast the problem as a set of linear equations. We have:

$$be^{cx_i} = d_i, \quad i = 0, \dots, n$$

compute the natural logarithm on both sides:

$$\underbrace{\ln b}_{a_0} + \underbrace{c}_{a_1} x_i = \underbrace{\ln d_i}_{f_i}$$

Now, we can apply a polynomial least squares fit to the problem, and once we have  $(a_0, a_1)$ ,  $b = e^{a_0}$  and  $c = a_1$ .

**Note:** This does **not** give the least squares fit to the original problem!!! (It gives us a pretty good estimate.)

## But... That is not a True Least Squares Fit!

**Note:** Fitting the modified problem does **not** give the least squares fit to the original problem!!!

In order to find the true least squares fit we need to know how to find roots and/or minima/maxima of non-linear systems of equations.

Feel free to sneak a peek at Burden-Faires chapter 10. Unfortunately we do not have the time to talk about this here...

What we need: **Math 693a — Numerical Optimization Techniques.**

Some of this stuff may show up in a different context in: **Math 562 — Mathematical Methods of Operations Research.**

## Cricket Data Analysis

C. A. Bessey and E. A. Bessey collected data on eight different crickets that they observed in Lincoln, Nebraska during August and September, 1897. The number of chirps/min was  $N$  and the temperature was  $T$ .

Create matrices

$$A_1 = \begin{pmatrix} 1 & N_1 \\ 1 & N_2 \\ \vdots & \vdots \end{pmatrix} \quad A_2 = \begin{pmatrix} 1 & N_1 & N_1^2 \\ 1 & N_2 & N_2^2 \\ \vdots & \vdots & \vdots \end{pmatrix}$$

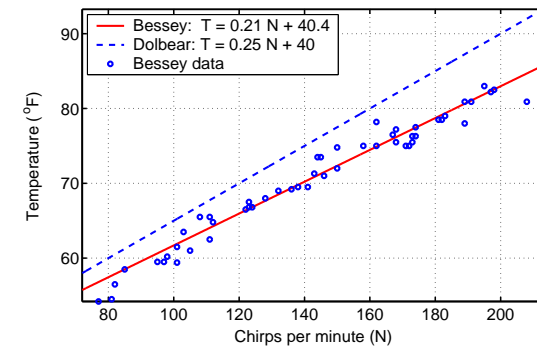
$$A_3 = \begin{pmatrix} 1 & N_1 & N_1^2 & N_1^3 \\ 1 & N_2 & N_2^2 & N_2^3 \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \quad A_4 = \begin{pmatrix} 1 & N_1 & N_1^2 & N_1^3 & N_1^4 \\ 1 & N_2 & N_2^2 & N_2^3 & N_2^4 \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

## Cricket Thermometer Application

Example source: Joe Mahaffy

There is a folk method of approximating the temperature (in Fahrenheit). This entered the scientific literature in 1896 by Dolbear with data collected by the Bessey brothers in 1898.

The temperature is approximated from the rate of crickets chirping by taking the number of chirps/min dividing by 4 and adding 40.



## Cricket Linear Model

*If you compute the matrix which you never should!*

$$A_1^T A_1 = \begin{pmatrix} 52 & 7447 \\ 7447 & 1133259 \end{pmatrix},$$

it has eigenvalues

$$\lambda_1 = 3.0633 \quad \text{and} \quad \lambda_2 = 1,133,308,$$

which gives the condition number

$$\text{cond}(A_1^T A_1) = 3.6996 \times 10^5.$$

Whereas

$$\text{cond}(A_1) = 608.2462.$$

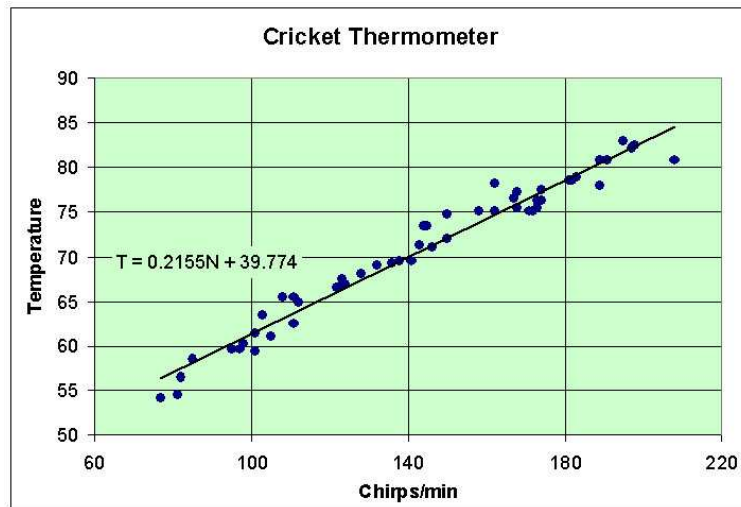
In Matlab

$$A_1 \backslash T$$

gives the parameters for best linear model

$$T_1(N) = 0.2155 N + 39.7441.$$

## Polynomial Fits to the Data: Linear



Linear Fit

## Cricket Quadratic Model

Similarly, the matrix

$$A_2^T A_2 = \begin{pmatrix} 52 & 7447 & 1133259 \\ 7447 & 1133259 & 1.8113 \times 10^8 \\ 1133259 & 1.8113 \times 10^8 & 3.0084 \times 10^{10} \end{pmatrix},$$

has eigenvalues

$$\lambda_1 = 0.1957, \quad \lambda_2 = 42,706, \quad \lambda_3 = 3.00853 \times 10^{10}$$

which gives the condition number

$$\text{cond}(A_2^T A_2) = 1.5371 \times 10^{11}.$$

Whereas,

$$\text{cond}(A_2) = 3.9206 \times 10^5,$$

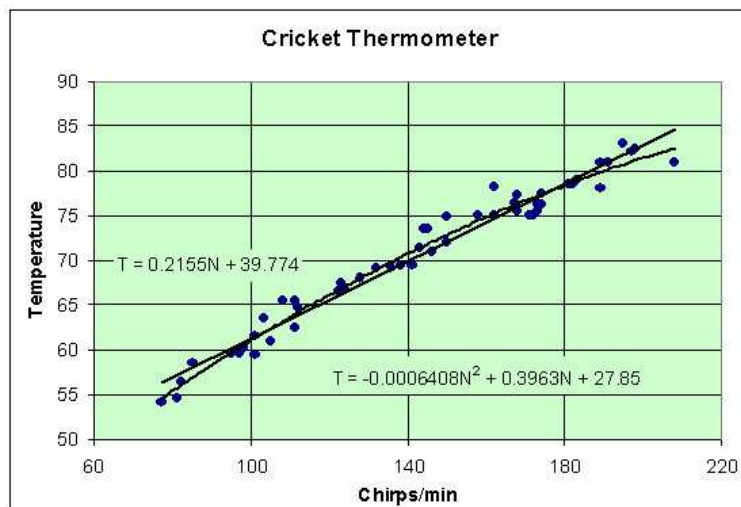
and

$$A_2 \setminus T,$$

gives the parameters for best quadratic model

$$T_2(N) = -0.00064076 N^2 + 0.39625 N + 27.8489.$$

## Polynomial Fits to the Data: Quadratic



Quadratic Fit

## Cricket Cubic and Quartic Models

The condition numbers for the cubic and quartic rapidly get larger with

$$\text{cond}(A_3^T A_3) = 6.3648 \times 10^{16} \quad \text{and} \quad \text{cond}(A_4^T A_4) = 1.1218 \times 10^{23}$$

These last two condition numbers suggest that any coefficients obtained are highly suspect.

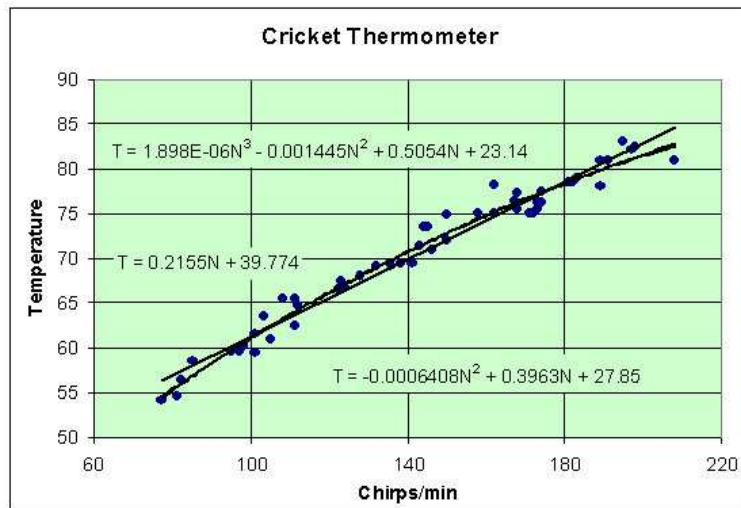
However, if done right, we are "only" subject to the condition numbers

$$\text{cond}(A_3) = 2.522 \times 10^8, \quad \text{cond}(A_4) = 1.738 \times 10^{11}.$$

The best cubic and quartic models are given by

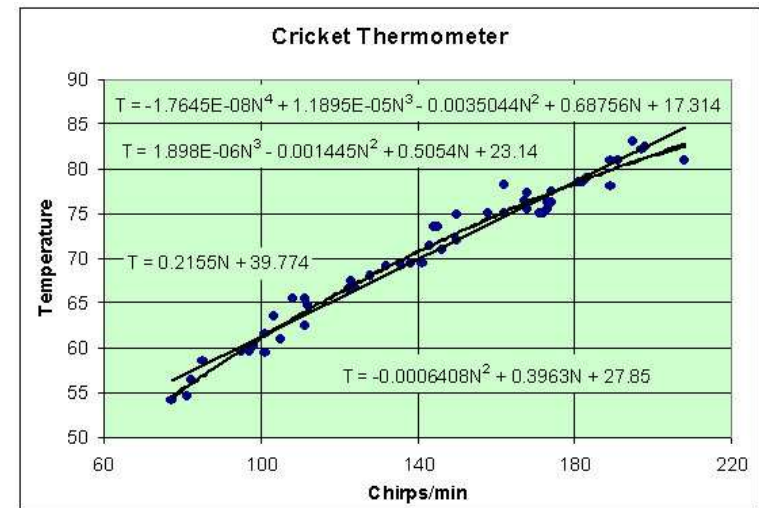
$$\begin{aligned} T_3(N) &= 0.0000018977 N^3 - 0.001445 N^2 + 0.50540 N + 23.138 \\ T_4(N) &= -0.00000001765 N^4 + 0.00001190 N^3 - 0.003504 N^2 \\ &= +0.6876 N + 17.314 \end{aligned}$$

## Polynomial Fits to the Data: Cubic



Cubic Fit

## Polynomial Fits to the Data: Quartic



Quartic Fit

## Best Cricket Model

So how does one select the best model?

Visually, one can see that the linear model does a very good job, and one only obtains a slight improvement with a quadratic. Is it worth the added complication for the slight improvement.

It is clear that the sum of square errors (SSE) will improve as the number of parameters increase.

From statistics, it is hotly debated how much penalty one should pay for adding parameters.

## Best Cricket Model - Analysis

## Bayesian Information Criterion

Let  $n$  be the number of data points,  $SSE$  be the sum of square errors, and let  $k$  be the number of parameters in the model.

$$BIC = n \ln(SSE/n) + k \ln(n).$$

## Akaike Information Criterion

$$AIC = 2k + n(\ln(2\pi SSE/n) + 1).$$



## Best Cricket Model - Analysis Continued

The table below shows the by the Akaike information criterion that we should take a quadratic model, while using a Bayesian Information Criterion we should use a cubic model.

	Linear	Quadratic	Cubic	Quartic
<i>SSE</i>	108.8	79.08	78.74	<b>78.70</b>
<i>BIC</i>	46.3	33.65	<b>33.43</b>	37.35
<i>AIC</i>	189.97	<b>175.37</b>	177.14	179.12

Returning to the original statement, we do fairly well by using the folk formula, despite the rest of this analysis!