Numerical Analysis and Computing

Lecture Notes #11.5 — Polynomials and Their Oscillations

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Outline

Polynomials, Polynomials, Polynomials Everywhere!

We have spent quite a bit of time dealing with polynomials:

- Computation
  - Horner's Method
  - Neville's Method
- Representation
  - Monomials $a_k\, x^k$
  - Lagrange Coefficients $f(x_k)\, L_{n,k}(x)$
  - Newton's Divided Differences $f[x_0, \dots, x_n] \prod_{m=0}^{n-1}(x - x_m)$
- Applications
  - Osculating Polynomials
  - Cubic Splines
  - Numerical Differentiation
  - Numerical Integration

Polynomials and Oscillations

From experience, and vigorous hand-waving we "know" that high order polynomials tend to have large oscillations.

We are not the first ones to notice this phenomenon... In 1901 Runge discovered that oscillations near the end points of an interval tended to grow with the order of the polynomial in certain cases; the effect is now known as **Runge's phenomenon**.

Runge vs. Weierstrass???

Runge's Phenomenon seems to contradict Weierstrass approximation theorem, which for $f \in C[a,b]$:

$$\lim_{n\to\infty} \left(\max_{x\in[a,b]} |f(x) - P_n(x)| \right) = 0.$$

Figure: RUNGE VS. WEIERSTRASS...

Bringing Some Clarity to the Issue...

Runge looked at interpolating the function

$$f(x) = \frac{1}{(1+x^2)}, \quad x \in [-5, 5]$$

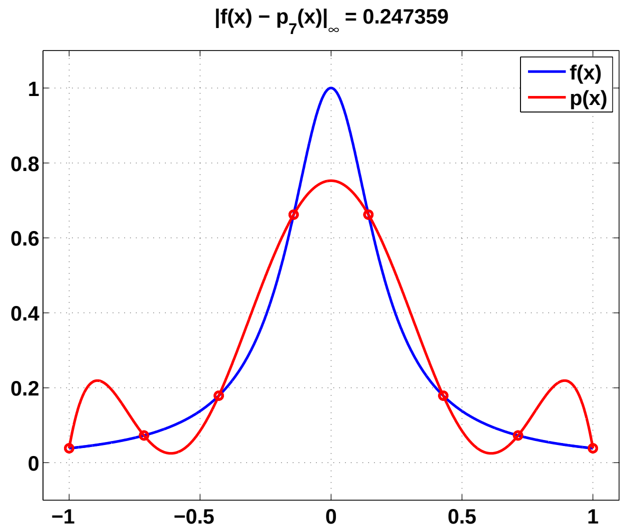
using equally spaced points; we start out by looking that the equivalent problem

$$f(x) = \frac{1}{(1+25\,x^2)}, \quad x \in [-1, 1],$$

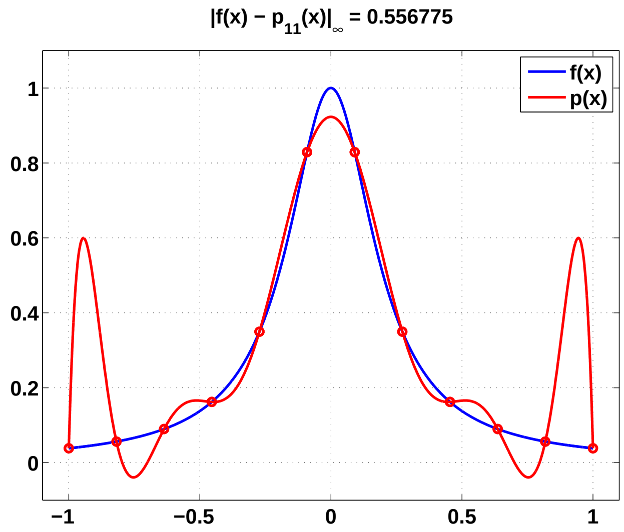
and then

$$f(x) = \frac{1}{(1+4\,x^2)}, \quad x \in [-1, 1].$$

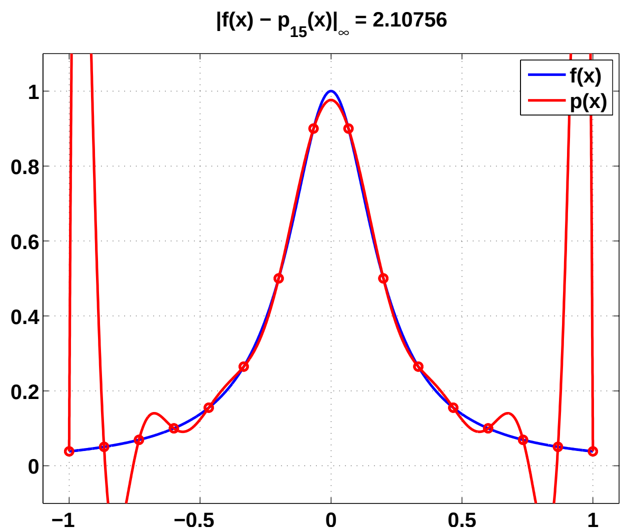
$P_7(x)$ Polynomial Interpolation of Runge's Function $f(x) = \frac{1}{1+25\,x^2}$



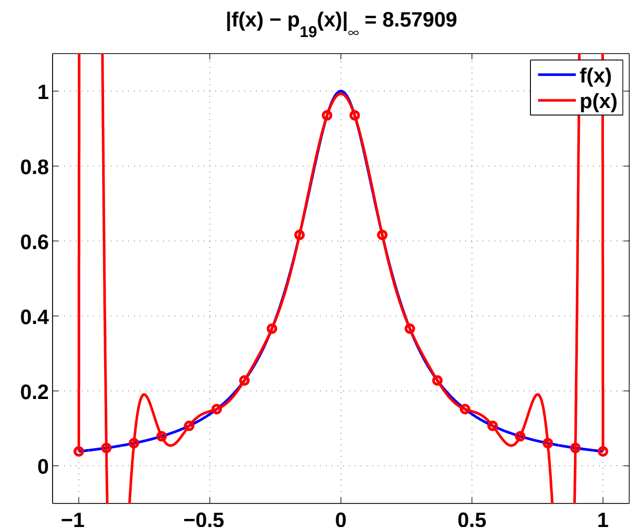
$P_{11}(x)$ Polynomial Interpolation of Runge's Function $f(x) = \frac{1}{1+25\,x^2}$



$P_{15}(x)$ Polynomial Interpolation of Runge's Function $f(x) = \frac{1}{1+25\,x^2}$

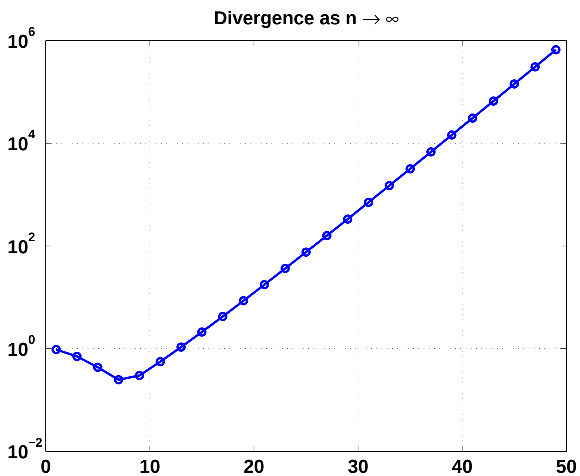


$P_{19}(x)$ Polynomial Interpolation of Runge's Function $f(x) = \frac{1}{1+25\,x^2}$



Convergence / Non-Convergence

We get exponential convergence in the middle of the interval, but exponential divergence near the ends. [Trefethen, 2013]



Lebesgue Function and Lebesgue Constant

The basic Lagrange formula for polynomial interpolation is given by

$$p_n(x) = \sum_{k=0}^{n} f_k L_{n,k}(x).$$

We use this representation on the interval $[-1, 1]$ and see how large $|p_n(x)|$ can get for a function with $|f_k| \leq 1$; this defines the **Lebesgue Function**

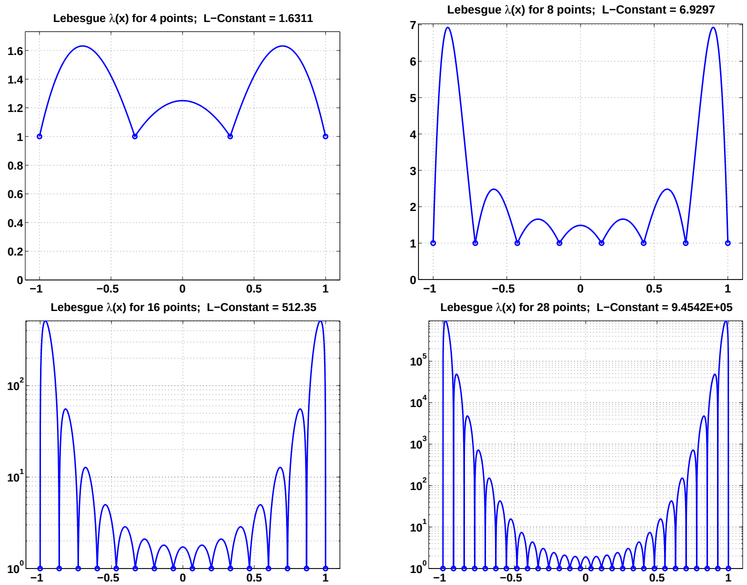
$$\lambda(x) = \sum_{k=0}^{n} |L_{n,k}(x)|,$$

its maximal value

$$\Lambda = \max_{x\in[-1,1]} \lambda(x)$$

is known as the **Lebesgue Constant**.

More Equi-Spaced Points ⇒ It Tends to Get Worse Examples



More Points ⇒ It Tends to Get Worse Theory

It is known that for equispaced points

$$\Lambda_n > \frac{2^{n-2}}{n^2}$$

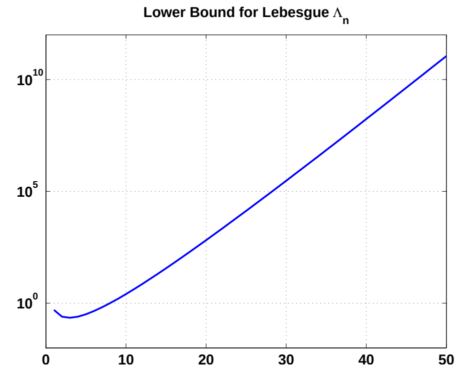
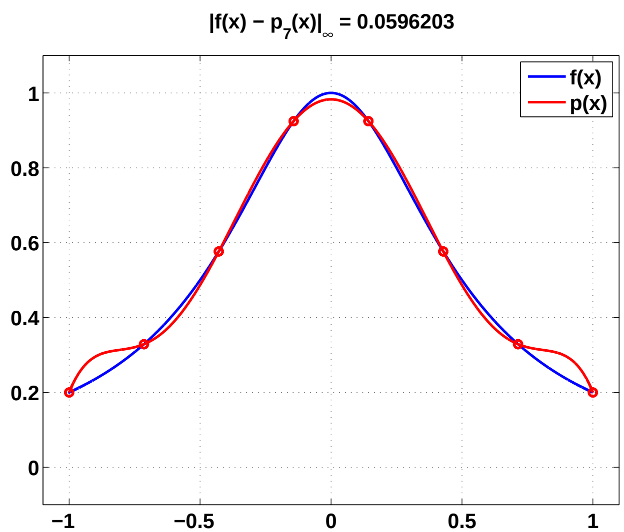
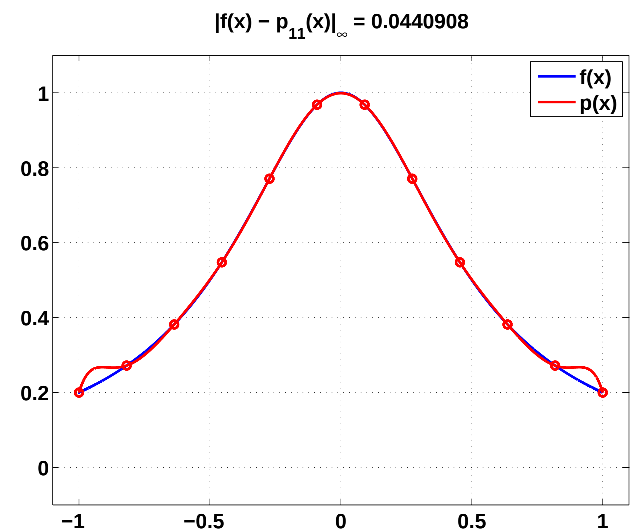


Figure: The lower bound for the worst-case scenario behavior of Λ. Notice that divergence for Runge's function followed this "shape," but didn't blow up as quickly.

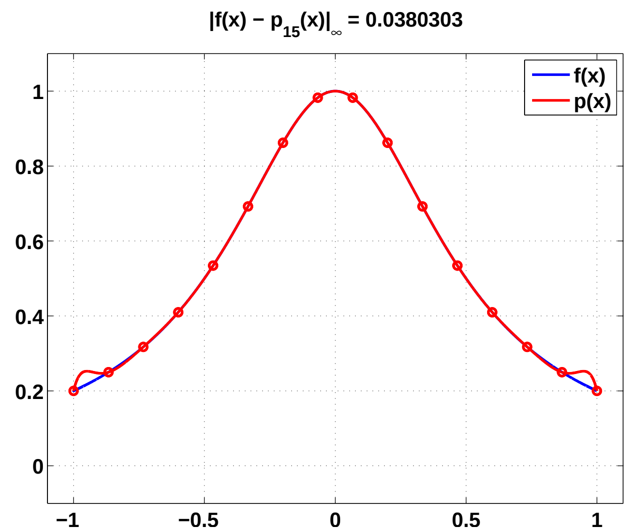
$P_7(x)$ Polynomial Interpolation of $f(x) = \frac{1}{1+4x^2}$



$P_{11}(x)$ Polynomial Interpolation of $f(x) = \frac{1}{1+4x^2}$



$P_{15}(x)$ Polynomial Interpolation of $f(x) = \frac{1}{1+4\,x^2}$



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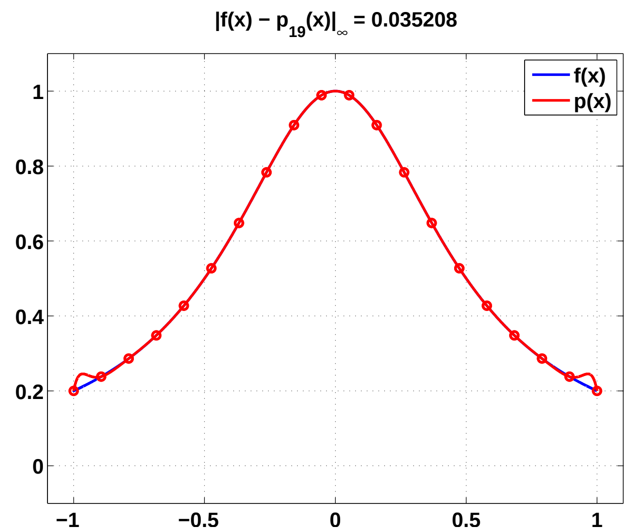
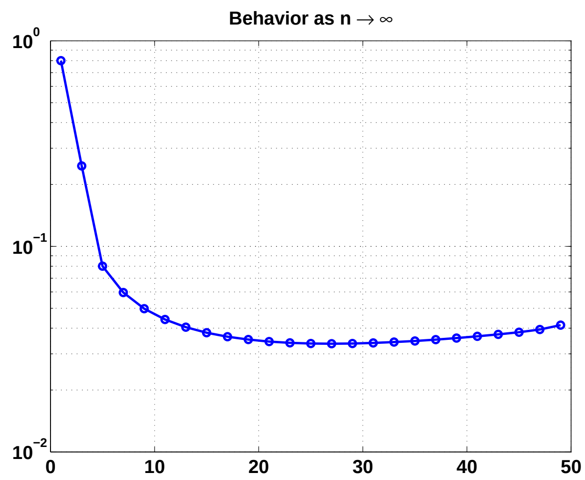
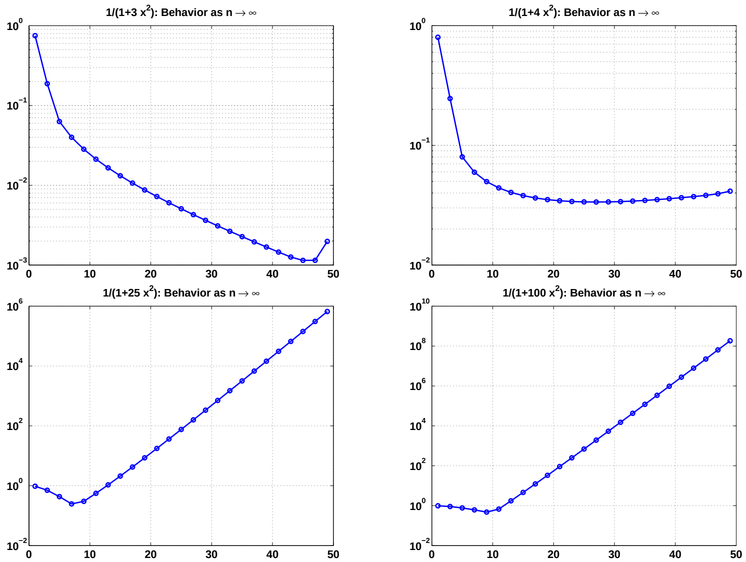


Convergence / Non-Convergence

Are we converging or diverging?



Convergence / Non-Convergence — A Few More Cases...



Explaining "Why" and "When" Oscillations Happen

We have seen examples of what can happen, and have the tools and language to quantify what is going on. However, we have not addressed "why?!" (under what circumstances) this (oscillations) happens...

... and is there some way to minimize / reduce the amount of oscillations???

In order to explain what is going on, we have to look beyond the scope of this class (we'll take a peek at it anyway, just for "fun!") and say something about potential theory in the complex plane.

$\ell(x)$

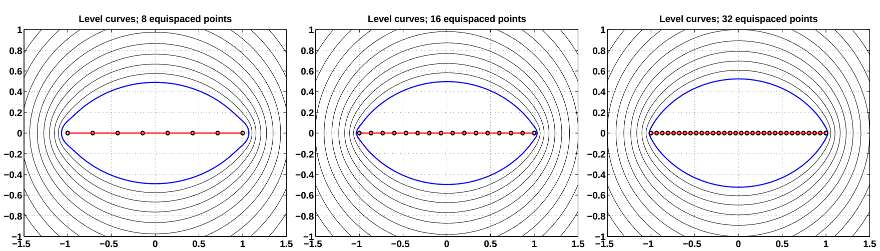
Let $\ell_n(x)$ be the polynomial with roots in the interpolation nodes $\{x_k\}_{k=0}^{n}$:

$$\ell_n(x) = \prod_{j=0}^{n}(x - x_j).$$

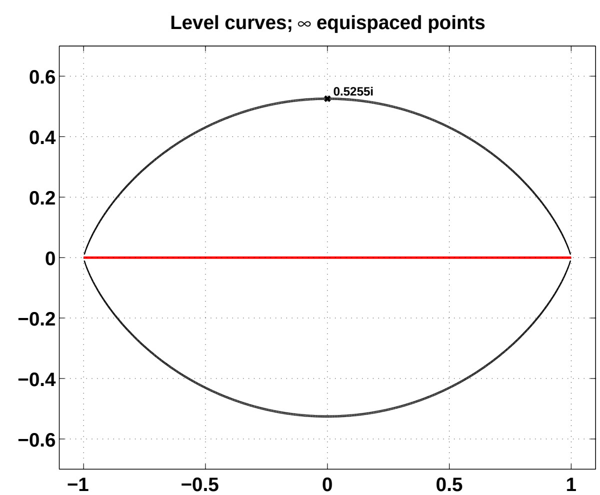
We notice (after some head-scratching) that we can express the Lagrange coefficients using $\ell_n(x)$:

$$L_{n,k}(x) = \frac{\ell_n(x)}{\ell_n'(x_k)(x - x_k)}$$

Level Curves of $|\ell_n(z)|$



The level curves of $|\ell_n(z)|$ in the complex plane matter; in particular, the level curve which wraps around the interval of interest $[-1, 1]$ as the number of interpolation points $n \nearrow \infty$ is important. For convergence, we need that the function we are interpolating is analytic (its Taylor series converges) everywhere inside that level curve — not only on the real interval $[-1, 1]$.



The critical level curve of $|\ell_\infty(z)|$ crosses the imaginary axis at $\approx \pm 0.5255i$.

Runge's Functions: Redux

We saw that interpolation of $\frac{1}{1+25x^2}$ was quite disastrous; now we understand why: the denominator has roots at $\pm 0.2\,i$, which means that the function has simple poles in those locations [and the Taylor expansion does not converge there].

For $\frac{1}{1+4x^2}$ the picture was not as clear, but now we can say with certainty that the interpolation will diverge as $n \to \infty$, since the function has poles at $\pm 0.5\,i$, just inside the critical level curve.

Function	$\frac{1}{1+3\,x^2}$	$\frac{1}{1+4\,x^2}$	$\frac{1}{1+25\,x^2}$	$\frac{1}{1+100\,x^2}$
Poles	$\pm\frac{1}{\sqrt{3}}\,i \approx \pm 0.577\,i$	$\pm 0.5\,i$	$\pm 0.2\,i$	$\pm 0.1\,i$
Location	Outside	Inside	Inside	Inside
Behavior	Convergence	Growing Oscillations	Growing Oscillations	Growing Oscillations

Looking Ahead

Now, Weierstrass promises that we can find excellent polynomial approximations to *any* function on *any* interval.

But! Clearly, equi-spaced interpolation can run into huge issues even for fairly nice-looking functions.

In the integration case, where we moved points around (Gaussian Quadrature) to optimize (maximize) the accuracy of the schemes... We can do something similar in the interpolation case: move the points so that we optimize (minimize) the onset of oscillations.

That's our next destination.