

# Numerical Analysis and Computing

## Lecture Notes #14

— Approximation Theory —

## Trigonometric Polynomial Approximation

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## Trigonometric Polynomials: A Very Brief History

$$P(x) = \sum_{n=0}^{\infty} a_n \cos(nx) + \sum_{n=0}^{\infty} b_n \sin(nx)$$

- 1750s Jean Le Rond d'Alembert used finite sums of  $\sin(nx)$  and  $\cos(nx)$  to study vibrations of a string.
- 17xx Use adopted by Leonhard Euler (leading mathematician at the time  $\Rightarrow$  validation for the approach).
- 17xx Daniel Bernoulli advocates use of **infinite** (as above) sums of sin and cos.
- 18xx **Jean Baptiste Joseph Fourier** used these infinite series to study heat flow. Developed theory.

## Fourier Series: First Observations

For each positive integer  $n$ , the set of functions  $\{\Phi_0, \Phi_1, \dots, \Phi_{2n-1}\}$ , where

$$\begin{cases} \Phi_0(x) = \frac{1}{2} \\ \Phi_k(x) = \cos(kx), & k = 1, \dots, n \\ \Phi_{n+k}(x) = \sin(kx), & k = 1, \dots, n-1 \end{cases}$$

is an **Orthogonal set** on the interval  $[-\pi, \pi]$  with respect to the weight function  $w(x) = 1$ .

## Orthogonality

Orthogonality follows from the fact that integrals over  $[-\pi, \pi]$  of  $\cos(kx)$  and  $\sin(kx)$  are zero (except  $\cos(0)$ ), and products can be rewritten as sums:

$$\left\{ \begin{array}{l} \sin \theta_1 \sin \theta_2 = \frac{\cos(\theta_1 - \theta_2) - \cos(\theta_1 + \theta_2)}{2} \\ \cos \theta_1 \cos \theta_2 = \frac{\cos(\theta_1 - \theta_2) + \cos(\theta_1 + \theta_2)}{2} \\ \sin \theta_1 \cos \theta_2 = \frac{\sin(\theta_1 - \theta_2) + \sin(\theta_1 + \theta_2)}{2}. \end{array} \right.$$

Let  $\mathcal{T}_n$  be the set of all linear combinations of the functions  $\{\Phi_0, \Phi_1, \dots, \Phi_{2n-1}\}$ ; this is the **set of trigonometric polynomials** of degree  $\leq n$ .

## The Fourier Series, $S(x)$

For  $f \in C[-\pi, \pi]$ , we seek the **continuous least squares approximation** by functions in  $\mathcal{T}_n$  of the form

$$S_n(x) = \frac{a_0}{2} + a_n \cos(nx) + \sum_{k=1}^{n-1} (a_k \cos(kx) + b_k \sin(kx)),$$

where, thanks to orthogonality

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx) dx, \quad b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx) dx.$$

Definition (Fourier Series)

The limit

$$S(x) = \lim_{n \rightarrow \infty} S_n(x)$$

is called the **Fourier Series** of  $f$ .

First we note that  $f(x)$  and  $\cos(kx)$  are even functions on  $[-\pi, \pi]$  and  $\sin(kx)$  are odd functions on  $[-\pi, \pi]$ . Hence,

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} |x| dx = \frac{2}{\pi} \int_0^{\pi} x dx = \pi.$$

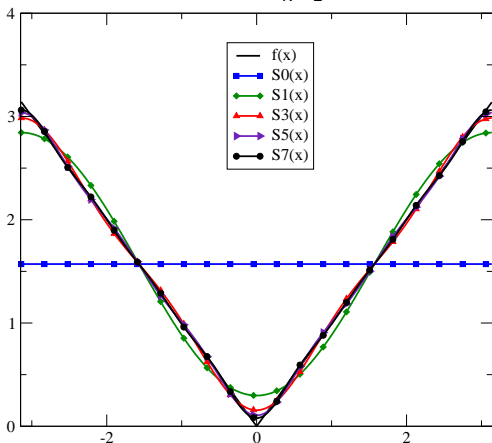
$$\begin{aligned} a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} |x| \cos(kx) dx = \frac{2}{\pi} \int_0^{\pi} x \cos(kx) dx \\ &= \underbrace{\frac{2}{\pi} x \frac{\sin(kx)}{k}}_0 \Big|_0^{\pi} - \frac{2}{k\pi} \int_0^{\pi} 1 \cdot \sin(kx) dx \\ &= \frac{2}{\pi k^2} [\cos(k\pi) - \cos(0)] = \frac{2}{\pi k^2} [(-1)^k - 1]. \end{aligned}$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \underbrace{|x| \sin(kx)}_{\text{even} \times \text{odd} = \text{odd}} dx = 0.$$

Example: Approximating  $f(x) = |x|$  on  $[-\pi, \pi]$

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We can write down  $S_n(x) = \frac{\pi}{2} + \frac{2}{\pi} \sum_{k=1}^n \frac{(-1)^k - 1}{k^2} \cos(kx)$



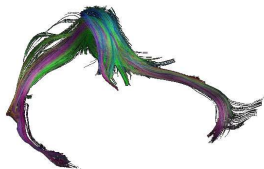


## The Discrete Fourier Transform: Introduction

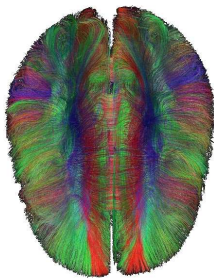
The discrete Fourier transform, a.k.a. the finite Fourier transform, is a transform on samples of a function.

It, and its “cousins,” are the most widely used mathematical transforms; applications include:

- Signal Processing
  - Image Processing
  - Audio Processing
- Data compression
- A tool for partial differential equations
- etc...



**Figure:** The fornix runs up from the hippocampus (an area important in memory formation) and ends in the hypothalamus (an area important in hunger and sleep regulation). **Credit:** Owen Philips (Google+, 18 April 2012).



**Figure:** Brain connectivity — the average connections of a group of people; our brains have largely the same underlying connections. **Credit:** Owen Philips (Google+, 3 April 2012).

## The Discrete Fourier Transform

Suppose we have  $2m$  data points,  $(x_j, f_j)$ , where

$$x_j = -\pi + \frac{j\pi}{m}, \text{ and } f_j = f(x_j), \quad j = 0, 1, \dots, 2m - 1.$$

The discrete least squares fit of a trigonometric polynomial  $S_n(x) \in \mathcal{T}_n$  minimizes

$$E(S_n) = \sum_{j=0}^{2m-1} [S_n(x_j) - f_j]^2.$$

## Orthogonality of the Basis Functions?

We know that the basis functions

$$\left\{ \begin{array}{l} \Phi_0(x) = \frac{1}{2} \\ \Phi_k(x) = \cos(kx), \quad k = 1, \dots, n \\ \Phi_{n+k}(x) = \sin(kx), \quad k = 1, \dots, n-1 \end{array} \right.$$

are orthogonal **with respect to integration over the interval**.

**The Big Question:** Are they orthogonal in the discrete case? Is the following true:

$$\sum_{j=0}^{2m-1} \Phi_k(x_j) \Phi_l(x_j) = \alpha_k \delta_{k,l} \quad ???$$

## Orthogonality of the Basis Functions! (A Lemma)...

Lemma

*If the integer  $r$  is not a multiple of  $2m$ , then*

$$\sum_{j=0}^{2m-1} \cos(rx_j) = \sum_{j=0}^{2m-1} \sin(rx_j) = 0.$$

*Moreover, if  $r$  is not a multiple of  $m$ , then*

$$\sum_{j=0}^{2m-1} [\cos(rx_j)]^2 = \sum_{j=0}^{2m-1} [\sin(rx_j)]^2 = m.$$

Recalling long-forgotten (or quite possible never seen) facts from **Complex Analysis — Euler's Formula**:

$$e^{i\theta} = \cos(\theta) + i \sin(\theta).$$

Thus,

$$\sum_{j=0}^{2m-1} \cos(rx_j) + i \sum_{j=0}^{2m-1} \sin(rx_j) = \sum_{j=0}^{2m-1} [\cos(rx_j) + i \sin(rx_j)] = \sum_{j=0}^{2m-1} e^{irx_j}.$$

Since

$$e^{irx_j} = e^{ir(-\pi + j\pi/m)} = e^{-ir\pi} e^{irj\pi/m},$$

we get

$$\sum_{j=0}^{2m-1} \cos(rx_j) + i \sum_{j=0}^{2m-1} \sin(rx_j) = e^{-ir\pi} \sum_{j=0}^{2m-1} e^{irj\pi/m}.$$

Since  $\sum_{j=0}^{2m-1} e^{irj\pi/m}$  is a **geometric series** with first term 1, and ratio  $e^{ir\pi/m} \neq 1$ , we get

$$\sum_{j=0}^{2m-1} e^{irj\pi/m} = \frac{1 - (e^{ir\pi/m})^{2m}}{1 - e^{ir\pi/m}} = \frac{1 - e^{2ir\pi}}{1 - e^{ir\pi/m}}.$$

This is zero since

$$1 - e^{2ir\pi} = 1 - \cos(2r\pi) - i \sin(2r\pi) = 1 - 1 - i \cdot 0 = 0.$$

This shows the first part of the lemma:

$$\sum_{j=0}^{2m-1} \cos(rx_j) = \sum_{j=0}^{2m-1} \sin(rx_j) = 0.$$

If  $r$  is not a multiple of  $m$ , then

$$\sum_{j=0}^{2m-1} [\cos(rx_j)]^2 = \sum_{j=0}^{2m-1} \frac{1 + \cos(2rx_j)}{2} = \sum_{j=0}^{2m-1} \frac{1}{2} = m.$$

Similarly (use  $\cos^2 \theta + \sin^2 \theta = 1$ )

$$\sum_{j=0}^{2m-1} [\sin(rx_j)]^2 = m.$$

This proves the second part of the lemma.

We are now ready to show that the basis functions are orthogonal.



## Showing Orthogonality of the Basis Functions

Recall

$$\left\{ \begin{array}{l} \sin \theta_1 \sin \theta_2 = \frac{\cos(\theta_1 - \theta_2) - \cos(\theta_1 + \theta_2)}{2} \\ \cos \theta_1 \cos \theta_2 = \frac{\cos(\theta_1 - \theta_2) + \cos(\theta_1 + \theta_2)}{2} \\ \sin \theta_1 \cos \theta_2 = \frac{\sin(\theta_1 - \theta_2) + \sin(\theta_1 + \theta_2)}{2}. \end{array} \right.$$

Thus for any pair  $k \neq l$

$$\sum_{j=0}^{2m-1} \Phi_k(x_j) \Phi_l(x_j)$$

is a zero-sum of sin or cos, and when  $k = l$ , the sum is  $m$ .

## Finally: The Trigonometric Least Squares Solution

Using

- [1] Our standard framework for deriving the least squares solution — set the partial derivatives with respect to all parameters equal to zero.
- [2] The orthogonality of the basis functions.

We find the coefficients in the summation

$$S_n(x) = \frac{a_0}{2} + a_n \cos(nx) + \sum_{k=1}^{n-1} (a_k \cos(kx) + b_k \sin(kx)) :$$

$$a_k = \frac{1}{m} \sum_{j=0}^{2m-1} f_j \cos(kx_j), \quad b_k = \frac{1}{m} \sum_{j=0}^{2m-1} f_j \sin(kx_j).$$

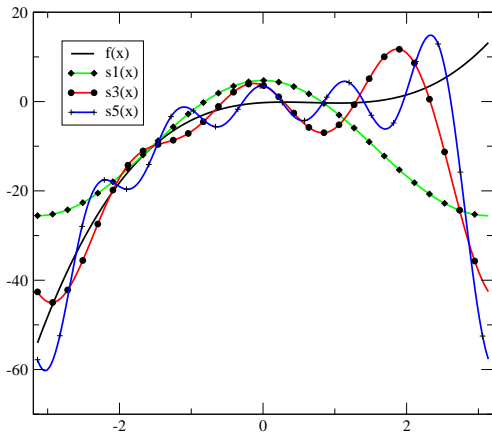
Let  $f(x) = x^3 - 2x^2 + x + 1/(x - 4)$  for  $x \in [-\pi, \pi]$ .

Let  $x_j = -\pi + j\pi/5$ ,  $j = 0, 1, \dots, 9$ , *i.e.*

| $j$ | $x_j$    | $f_j$     |
|-----|----------|-----------|
| 0   | -3.14159 | -54.02710 |
| 1   | -2.51327 | -31.17511 |
| 2   | -1.88495 | -15.85835 |
| 3   | -1.25663 | -6.58954  |
| 4   | -0.62831 | -1.88199  |
| 5   | 0        | -0.25     |
| 6   | 0.62831  | -0.20978  |
| 7   | 1.25663  | -0.28175  |
| 8   | 1.88495  | 1.00339   |
| 9   | 2.51327  | 5.08277   |

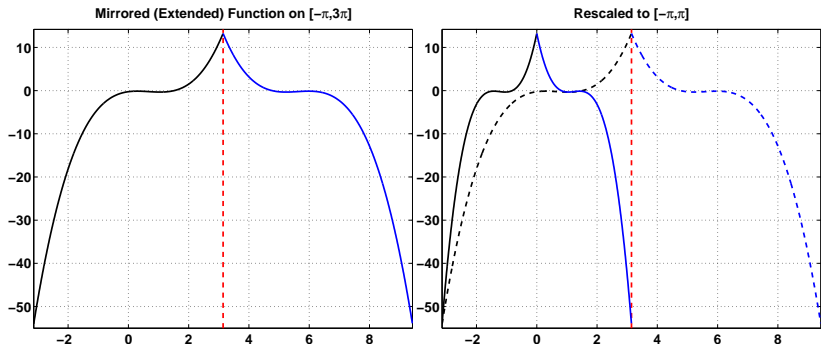
We get the following coefficients:

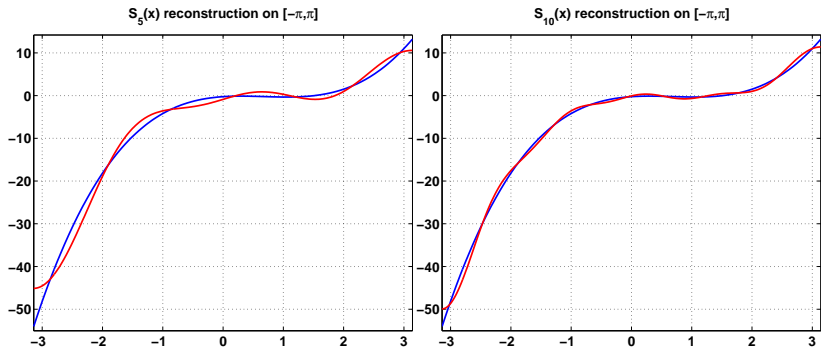
$$a_0 = -20.837, \quad a_1 = 15.1322, \quad a_2 = -9.0819, \quad a_3 = 7.9803$$
$$b_1 = 8.8661, \quad b_2 = -7.8193, \quad b_3 = 4.4910.$$



## Notes:

- [1] The approximation gets better as  $n \rightarrow \infty$ .
- [2] Since all the  $S_n(x)$  are  $2\pi$ -periodic, we will always have a problem when  $f(-\pi) \neq f(\pi)$ . [Fix: Periodic extension.] On the following two slides we see the performance for a  $2\pi$ -periodic  $f$ .
- [3] It seems like we need  $\mathcal{O}(m^2)$  operations to compute  $\tilde{\mathbf{a}}$  and  $\tilde{\mathbf{b}}$  —  $m$  sums, with  $m$  additions and multiplications. There is however a fast  $\mathcal{O}(m \log_2(m))$  algorithm that finds these coefficients. We will talk about this **Fast Fourier Transform** next time.





Let  $f(x) = 2x^2 + \cos(3x) + \sin(2x)$ ,  $x \in [-\pi, \pi]$ .

Let  $x_j = -\pi + j\pi/5$ ,  $j = 0, 1, \dots, 9$ , i.e.

| $j$ | $x_j$    | $f_j$   |
|-----|----------|---------|
| 0   | -3.14159 | 18.7392 |
| 1   | -2.51327 | 13.8932 |
| 2   | -1.88495 | 8.5029  |
| 3   | -1.25663 | 1.7615  |
| 4   | -0.62831 | -0.4705 |
| 5   | 0        | 1.0000  |
| 6   | 0.62831  | 1.4316  |
| 7   | 1.25663  | 2.9370  |
| 8   | 1.88495  | 7.3273  |
| 9   | 2.51327  | 11.9911 |



We get the following coefficients:

$$a_0 = -8.2685, \quad a_1 = 2.2853, \quad a_2 = -0.2064, \quad a_3 = 0.8729$$
$$b_1 = 0, \quad b_2 = 1, \quad b_3 = 0.$$

