Numerical Solutions to Differential Equations

Lecture Notes #3 — Euler's Method

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Outline

Euler's Method is a natural starting point for our discussion on numerical solutions of ODEs.

Usually the time points t_i are uniformly spaced, *i.e.* $t_i = t_0 + ih$. We write

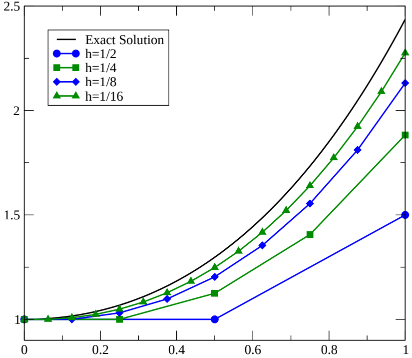
Euler's method

$$\mathbf{y_{i+1} = y_i + h\, f(t_i, y_i), \quad y(t_0) = y_0, \quad t_i = t_0 + i\, h}$$

Euler's Method — Example, $y' = y + 2t - 1;\ y(0) = 1$

Exact Solution: $y(t) = 2e^{t} - 2t - 1$

Euler's method on the interval $[0, 1]$, with $h \in \{1/2, 1/4, 1/8, 1/16\}$.



Accuracy:

We have seen that the **quality** of the numerical solution depends on the step size h.

Some of the concepts we need to define in order to analyze numerical methods for ODEs:

Consistency:

Is the numerical scheme solving the right problem?

Stability:

Is the numerical scheme robust with respect to propagation of round-off errors?

Convergence:

Do we get the right numerical solution as $h \to 0$???

Euler's Method: Derivation

Using our old ally (nemesis???) *(Math 541, or calculus)*, **Taylor's Theorem**, we can write:

$$y_{i+1} = y(t_{i+1}) = y(t_i + h) = y(t_i) + hy'(t_i) + \frac{h^2}{2} y''(\xi_i)$$
$$\xi_i \in [t_i, t_{i+1}]$$

We find that

$$\frac{y_{i+1} - y_i}{h} - y'(t_i) = \frac{h}{2} y''(\xi_i)$$

Approximation — Local Truncation Error (LTE).

Euler's method

$$y_{i+1} = y_i + h\,f(t_i)$$

is **consistent** with the differential equation

$$y'(t) = f(t)$$

since the Local Truncation Error satisfies

$$\lim_{h \to 0} \text{LTE}_{\text{Euler}}(h) = \lim_{h \to 0} \frac{h}{2} y''(\xi_i) = 0$$

A method is said to be of order p if

$$\lim_{h\to 0} \frac{\mathrm{LTE}(h)}{h^p} \leq \mathcal{C}$$

and

$$\lim_{h\to 0} \frac{\mathrm{LTE}(h)}{h^{p+\epsilon}} = \pm\infty, \quad \epsilon > 0.$$

Since $\mathrm{LTE}_{\mathrm{Euler}}(h) = \frac{h}{2} y''(\xi_i), \quad p_{\mathrm{Euler}} = 1.$

Euler's Method is a **first order method**.

A numerical method is said to be **unstable** if the error growth is exponential. — The total error depends on the local truncation error **and** the round-off error!

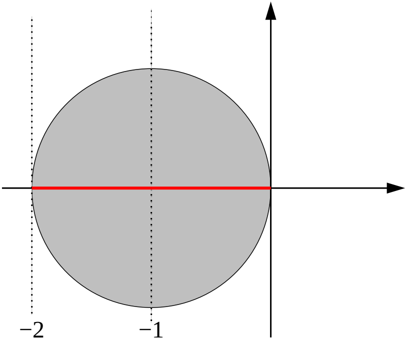
Consider the differential equation $y'(t) = \lambda y(t),\ y(t_0) = y_0$.

- The exact solution is given by $y(t) = y_0 e^{\lambda t}$.
- Euler's method applied to this problem:

$$y_{i+1} = (1 + h\lambda)y_i = (1 + h\lambda)^n y_0.$$

- This solution is stable **only if** $|1 + h\lambda| \leq 1$.

Region of Stability



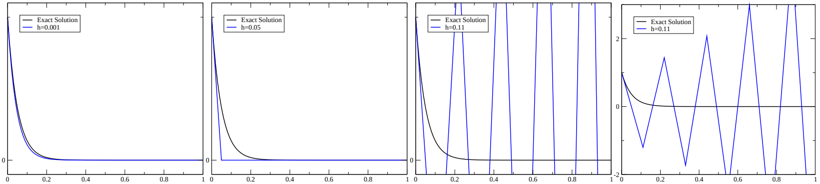
Euler's method is stable **only if** $|1 + h\lambda| \leq 1$. That is, $h\lambda$ must be inside the disk of radius 1, centered at -1 in the complex plane.

If λ is real $h\lambda$ must be in the interval $[-2, 0]$

Consider the ODE (exact solution $y(t) = e^{-20t}$)

$$y'(t) = -20y(t), \quad y(0) = 1$$

Since $\lambda = -20$, we must have $h < 0.1$ for stability...



$h = 0.001$ $h = 0.05$ $h = 0.11$ $h = 0.11$

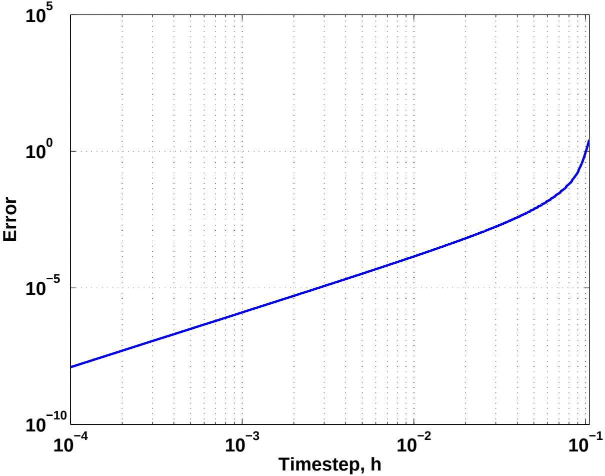
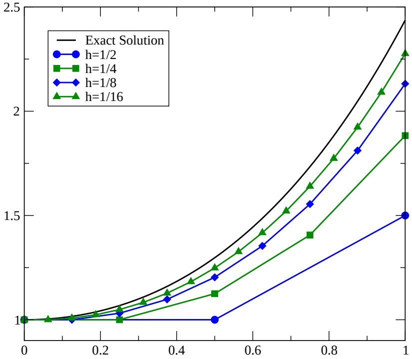


Figure: The size of the numerical error in the solution, plotted against the time step h.

A method is said to be **convergent** if as we decrease the step length ($h \to 0$) the numerical solution converges to the exact solution.

E.g.:



Theorem: Consistency + Stability $\Rightarrow$ Convergence.

Local Truncation Error, LTE(h)

The local error introduced by the discretization.

Accuracy

The order of accuracy is the largest integer p such that

$$\lim_{h\to 0} \frac{\text{LTE}(h)}{h^p} \leq \mathcal{C}$$

Consistency

A method is consistent if

$$\lim_{h\to 0} \text{LTE}(h) = 0$$

Stability

A scheme is unstable if it produces exponentially growing solutions for a problem for which the exact solution is bounded. Usually stability introduces **restrictions on the step size** h**.**

Region of Stability

The range of $h\lambda$ for which the selected method is stable.

Convergence

The numerical solution converges to the exact solution if the scheme is Consistent and Stable.

Euler's method applied to the system

$$\begin{cases} \frac{dy_1}{dt} = f_1(t, y_1, y_2, \ldots, y_n), & y_1(t_0) = y_{1,0} \\ \frac{dy_2}{dt} = f_2(t, y_1, y_2, \ldots, y_n), & y_2(t_0) = y_{2,0} \\ \vdots & \vdots \\ \frac{dy_n}{dt} = f_n(t, y_1, y_2, \ldots, y_n), & y_n(t_0) = y_{n,0} \end{cases}$$

is simply

$$\begin{cases} y_{1,i+1} = y_{1,i} + h\, f_1(t, y_1, y_2, \ldots, y_n) \\ y_{2,i+1} = y_{2,i} + h\, f_2(t, y_1, y_2, \ldots, y_n) \\ \vdots \\ y_{n,i+1} = y_{n,i} + h\, f_n(t, y_1, y_2, \ldots, y_n) \end{cases}$$

Beyond Euler's Method

Euler's method is easy to implement, but...

- The step-size h must be very small to achieve an acceptable level of accuracy (locally, the LTE).
- If we are solving over a long time period $[0, T]$ with small step-size, the method is expensive (requires many iterations) and slow.
- Local errors accumulate. The LTE $\sim \mathcal{O}(h)$ but we need $\sim 1/h$ iterations, in order to compute up to a fixed final time T. This could mean trouble?

Back to the Drawing Board — More Taylor Series...

Our first improvement of Euler's method is to keep more terms in the Taylor expansion

$$y(t_{i+1}) = \sum_{k=0}^{n} \frac{h^k}{k!} y^{(k)}(t_i) + \frac{h^{n+1}}{(n+1)!} y^{(n+1)}(\xi_i), \quad \xi_i \in [t_i, t_{i+1}]$$

The last term is the **remainder term** which corresponds to the **local truncation error**. Recall that for Euler's method we set $n = 1$, and ignored higher order terms.
From the differential equation

$$y'(t) = f(t, y), \quad y(t_0) = y_0$$

we can get expressions for higher order derivatives of y with the help of the **chain rule**.

$$y'(t) = f(t, y)$$

$$y''(t) = \frac{d[y'(t)]}{dt} = \frac{df(t, y)}{dt} = \frac{\partial f(t, y)}{\partial t} + \frac{\partial f(t, y)}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial f(t, y)}{\partial t} + \frac{\partial f(t, y)}{\partial y} f(t, y) \equiv f'(t, y)$$

We can continue in the same manner to get the relations

$$y^{(n)}(t) = f^{(n-1)}(t)...$$

With these expressions we write the *n*th order Taylor method as

$$y_{i+1} = y_i + \sum_{k=1}^{n} \frac{h^k}{k!} f^{(k-1)}(t_i, y_i).$$

Note that Euler's method is Taylor method of order 1.

Example: Higher Order Taylor Series Methods, I

We consider

$$y'(t) = y(t) + 2t - 1, \quad y(0) = 1.$$

We get

$$\begin{aligned}
f(t,y) &= y + 2t - 1 \\
f'(t,y) &= 2 + 1 \cdot (y + 2t - 1) = y + 2t + 1 \\
f''(t,y) &= 2 + 1 \cdot (y + 2t - 1) = y + 2t + 1 \\
&\vdots \\
f^{(n)}(t,y) &= y + 2t + 1
\end{aligned}$$

Example: Higher Order Taylor Series Methods, II

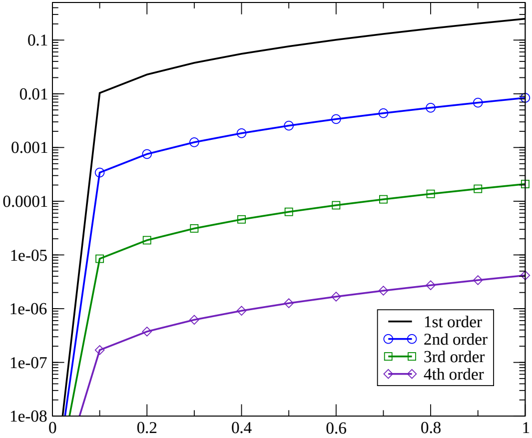


Figure: The **error** (on a logarithmic scale) for 1st, 2nd, 3rd and 4th order Taylor methods applied to $y' = y + 2t - 1,\ y(0) = 1$ on the interval $[0, 1]$ with step size $h = 0.1$.

Example: Higher Order Taylor Series Methods, II$\frac{1}{2}$

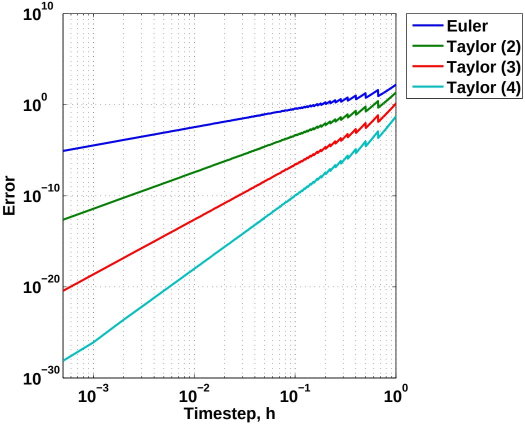


Figure: The **error** (on a log-log scale) for 1st, 2nd, 3rd and 4th order Taylor methods applied to $y' = y + 2t - 1, \; y(0) = 1$ on the interval $[0, 2]$ with step size $h \in [10^{-4}, 1]$.

Multi-Point Methods

In Euler's method $f(t, y)$ is computed at the beginning of the interval $[t_i, t_{i+1}]$, and then assumed (approximated) to be constant over the interval.

In Taylor's method $f^{(k)}(t, y)$, $k = 0, 1, \ldots, n$ are computed at the beginning of the interval, and then assumed (approximated) to be constant.

Reminiscent to the polynomial interpolation in **Math 541**, we will now introduce a multi-point method — where the derivative(s) is(are) computed at more than one point.

Heun's method is a simple version of a **predictor-corrector method** (this topic will be revisited in great detail later):

1. Start with the initial condition $y(t_0) = y_0$.
2. In order to compute y_{i+1}:
    1. Use Euler's method as the predictor:

    $$y_{i+1}^0 = y_i + hf(t_i, y_i)$$

    2. Use the slope at the end-point as the **corrector**:

    $$y_{i+1} = y_i + h\left[\frac{f(t_i, y_i) + f(t_{i+1}, y_{i+1}^0)}{2}\right]$$

Note that the corrective step can be repeated.

Heun's Method: Error Analysis

Taylor expand

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2}y_i'' + \frac{h^3}{6}y'''(\xi_i)$$

Approximate

$$y_i'' = \frac{y_{i+1}' - y_i'}{h} + \mathcal{O}(h)$$

Plug in...

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2}\left[\frac{y_{i+1}' - y_i'}{h}\right] + \mathcal{O}(h^3)$$

Simplify

$$y_{i+1} = y_i + h\left[\frac{y_{i+1}' + y_i'}{2}\right] + \mathcal{O}(h^3)$$

Heun's Method: Error Analysis, II

Identify

$$y_{i+1} = y_i + h\left[\frac{f(t_{i+1}, y_{i+1}) + f(t_i, y_i)}{2}\right] + \mathcal{O}(h^3)$$

We have

$$y_{i+1} = y_i + h\left[\frac{f(t_{i+1}, y_{i+1}) + f(t_i, y_i)}{2}\right] + \mathcal{O}(h^3)$$

Hence,

$$\text{LTE}_{\text{Heun}}(h) = \frac{y_{i+1} - y_i}{h} - \left[\frac{f(t_{i+1}, y_{i+1}) + f(t_i, y_i)}{2}\right] = \mathcal{O}(h^2)$$

In summary: Heun's Method is a **second order** method.

A "midpoint-rulish" modification of Euler's method:

Take a half-step using Euler's method

$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(t_i, y_i)$$

Now, compute the slope at this center-point:

$$y'_{i+\frac{1}{2}} = f\left(t_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}\right)$$

Use this slope as an approximation of the slope throughout the interval $[t_i, t_{i+1}]$:

$$y_{i+1} = y_i + h\, f\left(t_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}\right)$$

For this version of Euler's method LTE $\sim \mathcal{O}(h^2)$ (HW#1).

Find the numerical solution of the problem

$$y' = -5y + e^{-2t}, \quad y(0) = 1$$

in the interval $[0, 1]$ using:

1. Euler's Method
2. Taylor's ($n = 2$) Method
3. Heun's Method

in the interval $[0, 1]$, with $h = 0.05$, and $h = 0.025$.

Submit: Code, and plots of your solutions.

Note that the exact solution is $\mathbf{y(t) = \frac{1}{3}\left(e^{-2t} + 2e^{-5t}\right)}$.

Show that the "Midpoint-rulish" version of Euler's method is second order.

Find **expressions** for the **regions of stability** for Taylor's method of orders 2 and 3, for the equation

$$y' = \lambda y.$$

What are the entire regions of stability?