Numerical Solutions to Differential Equations

Lecture Notes #3 — Euler's Method

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Outline

Euler's Method

Euler's Method is a natural starting point for our discussion on numerical solutions of ODEs.

Usually the time points t_i are uniformly spaced, *i.e.* $t_i = t_0 + ih$. We write

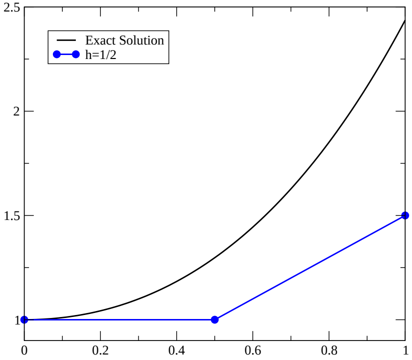
Euler's method

$$\mathbf{y_{i+1} = y_i + h\,f(t_i, y_i), \quad y(t_0) = y_0, \quad t_i = t_0 + i\,h}$$

Euler's Method — Example, $y' = y + 2t - 1;\ y(0) = 1$

Exact Solution: $y(t) = 2e^{t} - 2t - 1$

Euler's method on the interval $[0, 1]$, with $h = 1/2$.



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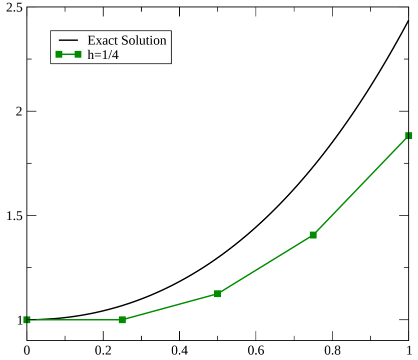

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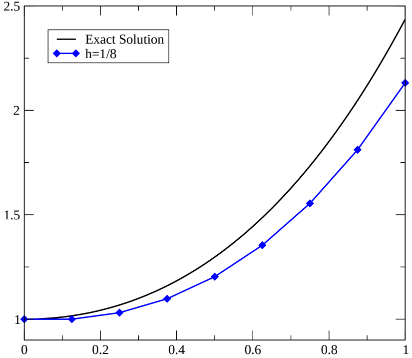

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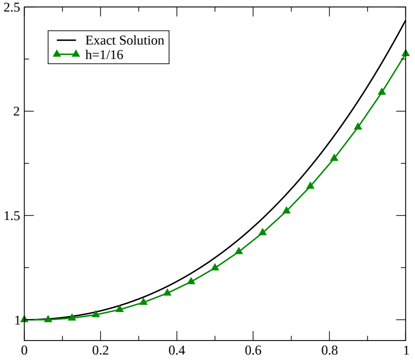
Euler's method on the interval $[0, 1]$, with $h = 1/16$.



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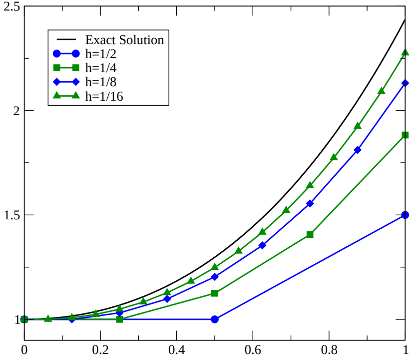
Euler's method on the interval $[0, 1]$, with $h \in \{1/2, 1/4, 1/8, 1/16\}$.



Euler's Method — Things to Quantify

Accuracy:

We have seen that the **quality** of the numerical solution depends on the step size h.

Some of the concepts we need to define in order to analyze numerical methods for ODEs:

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Accuracy:

We have seen that the **quality** of the numerical solution depends on the step size h.

Some of the concepts we need to define in order to analyze numerical methods for ODEs:

Consistency:

Is the numerical scheme solving the right problem?

Stability:

Is the numerical scheme robust with respect to propagation of round-off errors?

Convergence:

Do we get the right numerical solution as $h \to 0$???

Euler's Method: Derivation

Using our old ally (nemesis???) *(Math 541, or calculus)*, **Taylor's Theorem**, we can write:

$$y_{i+1} = y(t_{i+1}) = y(t_i + h) = y(t_i) + hy'(t_i) + \frac{h^2}{2} y''(\xi_i)$$
$$\xi_i \in [t_i, t_{i+1}]$$

We find that

$$\frac{y_{i+1} - y_i}{h} - y'(t_i) = \frac{h}{2} y''(\xi_i)$$

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Approximation

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Approximation — **Local Truncation Error (LTE)**.

Consistency

Euler's method

$$y_{i+1} = y_i + h\, f(t_i)$$

is **consistent** with the differential equation

$$y'(t) = f(t)$$

since the Local Truncation Error satisfies

$$\lim_{h\to 0} \text{LTE}_{\text{Euler}}(h) = \lim_{h\to 0} \frac{h}{2} y''(\xi_i) = 0$$

Accuracy

A method is said to be of order p if

$$\lim_{h\to 0} \frac{\text{LTE}(h)}{h^p} \leq \mathcal{C}$$

and

$$\lim_{h\to 0} \frac{\text{LTE}(h)}{h^{p+\epsilon}} = \pm\infty, \quad \epsilon > 0.$$

Since $\text{LTE}_{\text{Euler}}(h) = \frac{h}{2} y''(\xi_i), \quad p_{\text{Euler}} = 1.$

Euler's Method is a **first order method**.

Stability

A numerical method is said to be **unstable** if the error growth is exponential. — The total error depends on the local truncation error **and** the round-off error!

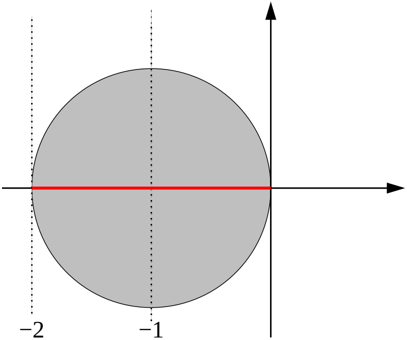
Consider the differential equation $y'(t) = \lambda y(t)$, $y(t_0) = y_0$.

- The exact solution is given by $y(t) = y_0 e^{\lambda t}$.
- Euler's method applied to this problem:

$$y_{i+1} = (1 + h\lambda)y_i = (1 + h\lambda)^n y_0.$$

- This solution is stable **only if** $|1 + h\lambda| \leq 1$.

Region of Stability



Euler's method is stable **only if** $|1 + h\lambda| \leq 1$. That is, $h\lambda$ must be inside the disk of radius 1, centered at -1 in the complex plane.

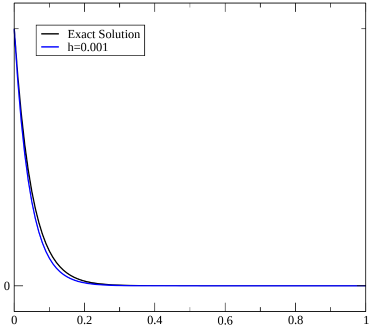
If λ is real $h\lambda$ must be in the interval $[-2, 0]$

Stability: Example (∃ Movie)

Consider the ODE (exact solution $y(t) = e^{-20t}$)

$$y'(t) = -20y(t), \quad y(0) = 1$$

Since $\lambda = -20$, we must have $h < 0.1$ for stability...



$h = 0.001$

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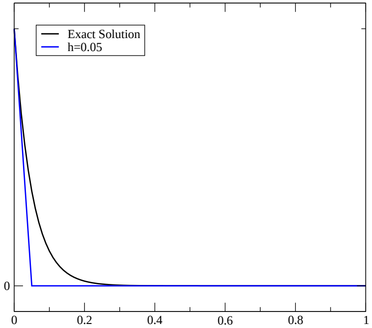

$h = 0.05$

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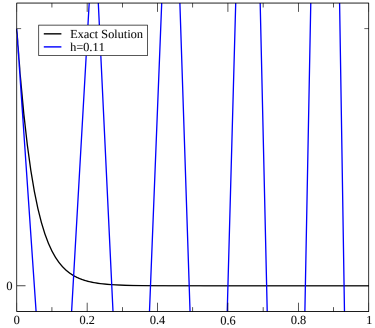

$h = 0.11$

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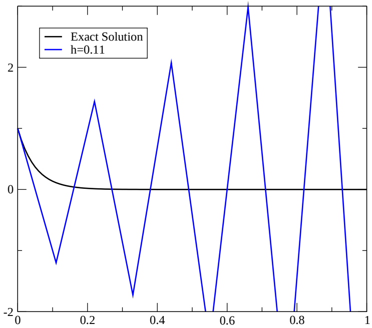

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Stability: Example

Error Plot

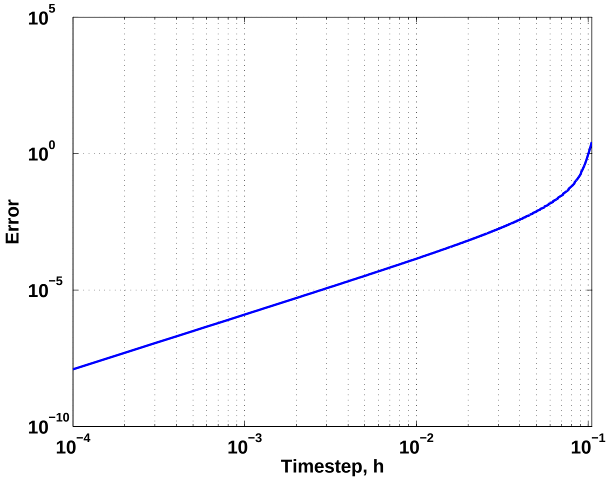
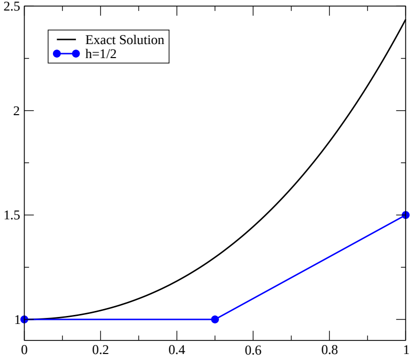


Figure: The size of the numerical error in the solution, plotted against the time step h.

Convergence

A method is said to be **convergent** if as we decrease the step length ($h \to 0$) the numerical solution converges to the exact solution.

E.g.:



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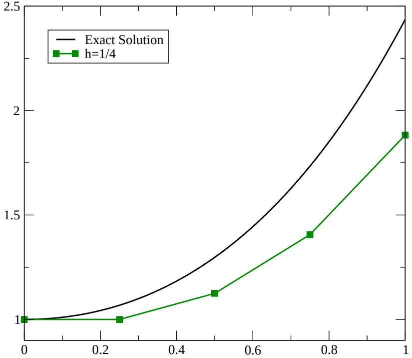

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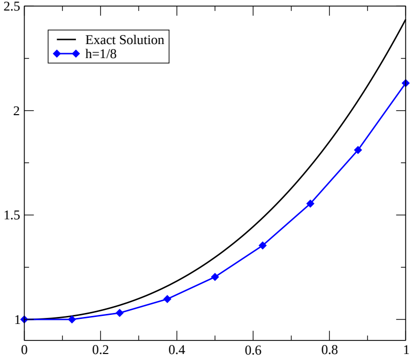

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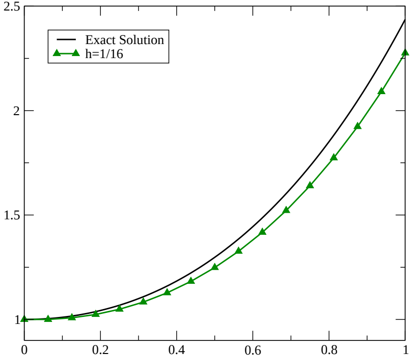

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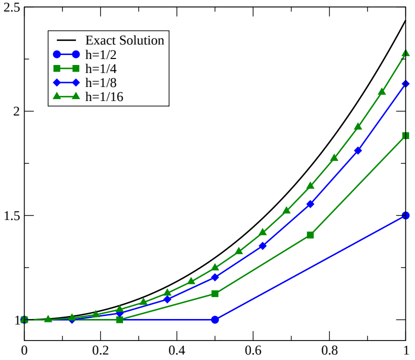

Theorem: Consistency + Stability $\Rightarrow$ Convergence.

Summary: Key Concepts Introduced

Local Truncation Error, LTE(h)

The local error introduced by the discretization.

Accuracy

The order of accuracy is the largest integer p such that

$$\lim_{h \to 0} \frac{\text{LTE}(h)}{h^p} \leq \mathcal{C}$$

Consistency

A method is consistent if

$$\lim_{h \to 0} \text{LTE}(h) = 0$$

Summary: Key Concepts Introduced, II

Stability

A scheme is unstable if it produces exponentially growing solutions for a problem for which the exact solution is bounded. Usually stability introduces **restrictions on the step size h.**

Region of Stability

The range of $h\lambda$ for which the selected method is stable.

Convergence

The numerical solution converges to the exact solution if the scheme is Consistent and Stable.

Euler's Method for Systems of ODEs

Euler's method applied to the system

$$\begin{cases} \frac{dy_1}{dt} = f_1(t, y_1, y_2, \ldots, y_n), & y_1(t_0) = y_{1,0} \\ \frac{dy_2}{dt} = f_2(t, y_1, y_2, \ldots, y_n), & y_2(t_0) = y_{2,0} \\ \vdots & \vdots \\ \frac{dy_n}{dt} = f_n(t, y_1, y_2, \ldots, y_n), & y_n(t_0) = y_{n,0} \end{cases}$$

is simply

$$\begin{cases} y_{1,i+1} = y_{1,i} + h\, f_1(t, y_1, y_2, \ldots, y_n) \\ y_{2,i+1} = y_{2,i} + h\, f_2(t, y_1, y_2, \ldots, y_n) \\ \vdots \\ y_{n,i+1} = y_{n,i} + h\, f_n(t, y_1, y_2, \ldots, y_n) \end{cases}$$

Beyond Euler's Method

Euler's method is easy to implement, but...

- The step-size h must be very small to achieve an acceptable level of accuracy (locally, the LTE).
- If we are solving over a long time period $[0, T]$ with small step-size, the method is expensive (requires many iterations) and slow.
- Local errors accumulate. The LTE $\sim \mathcal{O}(h)$ but we need $\sim 1/h$ iterations, in order to compute up to a fixed final time T. This could mean trouble?

Back to the Drawing Board — More Taylor Series...

Our first improvement of Euler's method is to keep more terms in the Taylor expansion

$$y(t_{i+1}) = \sum_{k=0}^{n} \frac{h^k}{k!} y^{(k)}(t_i) + \frac{h^{n+1}}{(n+1)!} y^{(n+1)}(\xi_i), \quad \xi_i \in [t_i, t_{i+1}]$$

The last term is the **remainder term** which corresponds to the **local truncation error**.

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The last term is the **remainder term** which corresponds to the **local truncation error**. Recall that for Euler's method we set $n = 1$, and ignored higher order terms.
From the differential equation

$$y'(t) = f(t, y), \quad y(t_0) = y_0$$

we can get expressions for higher order derivatives of y with the help of the **chain rule**.

Lost Calculus Treasures™: Applying the Chain Rule

$$y'(t) = f(t,y)$$

$$y''(t) = \frac{d[y'(t)]}{dt} = \frac{df(t,y)}{dt} = \frac{\partial f(t,y)}{\partial t} + \frac{\partial f(t,y)}{\partial y}\frac{dy}{dt}$$

$$= \frac{\partial f(t,y)}{\partial t} + \frac{\partial f(t,y)}{\partial y} f(t,y) \equiv f'(t,y)$$

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We can continue in the same manner to get the relations

$$y^{(n)}(t) = f^{(n-1)}(t)\ldots$$

With these expressions we write the nth order Taylor method as

$$y_{i+1} = y_i + \sum_{k=1}^{n} \frac{h^k}{k!} f^{(k-1)}(t_i, y_i).$$

Note that Euler's method is Taylor method of order 1.

Example: Higher Order Taylor Series Methods, I

We consider

$$y'(t) = y(t) + 2t - 1, \quad y(0) = 1.$$

We get

$$\begin{aligned} f(t,y) &= y + 2t - 1 \\ f'(t,y) &= 2 + 1 \cdot (y + 2t - 1) = y + 2t + 1 \\ f''(t,y) &= 2 + 1 \cdot (y + 2t - 1) = y + 2t + 1 \\ &\vdots \\ f^{(n)}(t,y) &= y + 2t + 1 \end{aligned}$$

Example: Higher Order Taylor Series Methods, II

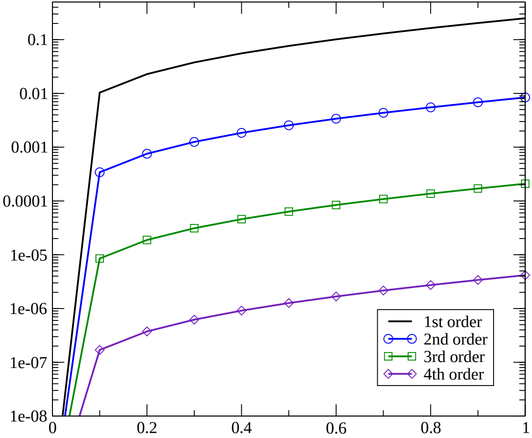

Figure: The **error** (on a logarithmic scale) for 1st, 2nd, 3rd and 4th order Taylor methods applied to $y' = y + 2t - 1, \;\; y(0) = 1$ on the interval $[0, 1]$ with step size $h = 0.1$.

Example: Higher Order Taylor Series Methods, II$\frac{1}{2}$

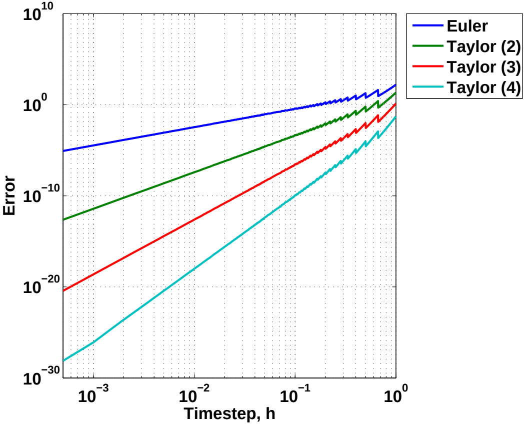

Figure: The **error** (on a log-log scale) for 1st, 2nd, 3rd and 4th order Taylor methods applied to $y' = y + 2t - 1, \; y(0) = 1$ on the interval $[0, 2]$ with step size $h \in [10^{-4}, 1]$.

Multi-Point Methods

In Euler's method $f(t, y)$ is computed at the beginning of the interval $[t_i, t_{i+1}]$, and then assumed (approximated) to be constant over the interval.

In Taylor's method $f^{(k)}(t, y)$, $k = 0, 1, \ldots, n$ are computed at the beginning of the interval, and then assumed (approximated) to be constant.

Reminiscent to the polynomial interpolation in **Math 541**, we will now introduce a multi-point method — where the derivative(s) is(are) computed at more than one point.

Heun's Method

Heun's method is a simple version of a **predictor-corrector method** (this topic will be revisited in great detail later):

1. Start with the initial condition $y(t_0) = y_0$.
2. In order to compute y_{i+1}:
   1. Use Euler's method as the predictor:

   $$y_{i+1}^{0} = y_i + hf(t_i, y_i)$$

   2. Use the slope at the end-point as the **corrector**:

   $$y_{i+1} = y_i + h\left[\frac{f(t_i, y_i) + f(t_{i+1}, y_{i+1}^{0})}{2}\right]$$

Note that the corrective step can be repeated.

Heun's Method: Error Analysis

Taylor expand

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2}y_i'' + \frac{h^3}{6}y'''(\xi_i)$$

Heun's Method: Error Analysis

Taylor expand

$$y_{i+1} = y_i + h y_i' + \frac{h^2}{2} y_i'' + \frac{h^3}{6} y'''(\xi_i)$$

Approximate

$$y_i'' = \frac{y_{i+1}' - y_i'}{h} + \mathcal{O}(h)$$

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Plug in...

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2}\left[\frac{y_{i+1}' - y_i'}{h}\right] + \mathcal{O}(h^3)$$

Heun's Method: Error Analysis

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Approximate

$$y_i'' = \frac{y_{i+1}' - y_i'}{h} + \mathcal{O}(h)$$

Plug in...

$$y_{i+1} = y_i + hy_i' + \frac{h^2}{2}\left[\frac{y_{i+1}' - y_i'}{h}\right] + \mathcal{O}(h^3)$$

Simplify

$$y_{i+1} = y_i + h\left[\frac{y_{i+1}' + y_i'}{2}\right] + \mathcal{O}(h^3)$$

Heun's Method: Error Analysis, II

Identify

$$y_{i+1} = y_i + h\left[\frac{f(t_{i+1}, y_{i+1}) + f(t_i, y_i)}{2}\right] + \mathcal{O}(h^3)$$

We have

$$y_{i+1} = y_i + h\left[\frac{f(t_{i+1}, y_{i+1}) + f(t_i, y_i)}{2}\right] + \mathcal{O}(h^3)$$

Hence,

$$\text{LTE}_{\text{Heun}}(h) = \frac{y_{i+1} - y_i}{h} - \left[\frac{f(t_{i+1}, y_{i+1}) + f(t_i, y_i)}{2}\right] = \mathcal{O}(h^2)$$

In summary: **Heun's Method** is a **second order** method.

Modified Euler's Method

A "midpoint-rulish" modification of Euler's method:

Take a half-step using Euler's method

$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(t_i, y_i)$$

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Take a half-step using Euler's method

$$y_{i+\frac{1}{2}} = y_i + \frac{h}{2} f(t_i, y_i)$$

Now, compute the slope at this center-point:

$$y'_{i+\frac{1}{2}} = f\left(t_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}\right)$$

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Now, compute the slope at this center-point:

$$y'_{i+\frac{1}{2}} = f\left(t_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}\right)$$

Use this slope as an approximation of the slope throughout the interval $[t_i, t_{i+1}]$:

$$y_{i+1} = y_i + h\, f\left(t_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}\right)$$

For this version of Euler's method LTE $\sim \mathcal{O}(h^2)$ (HW#1).

Homework #1 — Due 11:00am, 2/6/2015, I/II

Find the numerical solution of the problem

$$y' = -5y + e^{-2t}, \quad y(0) = 1$$

in the interval $[0, 1]$ using:

1. Euler's Method
2. Taylor's ($n = 2$) Method
3. Heun's Method

in the interval $[0, 1]$, with $h = 0.05$, and $h = 0.025$.

Submit: Code, and plots of your solutions.

Note that the exact solution is $\mathbf{y(t) = \frac{1}{3}\left(e^{-2t} + 2e^{-5t}\right)}$.

Homework #1 — Due 11:00am, 2/6/2015, II/II

Show that the "Midpoint-rulish" version of Euler's method is second order.

Find **expressions** for the **regions of stability** for Taylor's method of orders 2 and 3, for the equation

$$y' = \lambda y.$$

What are the entire regions of stability?