Numerical Solutions to Differential Equations

Lecture Notes #13
The Van der Pol Oscillator

Peter Blomgren,
⟨blomgren.peter@gmail.com⟩

Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

http://terminus.sdsu.edu/

Spring 2015

Outline

1. The Van der Pol Oscillator
   - Second order ODE $\rightsquigarrow$ 2D system
   - 2D-system $\rightsquigarrow$ Lienard Equation
2. Return to Physics — Circuit Analysis
   - R-C-L Circuit
3. The Van der Pol Oscillator, again...
   - (Physical) Stability Analysis of the Origin
   - Finally, Computations

The van der Pol oscillator was originally "discovered" by the Dutch electrical engineer and physicist Balthasar van der Pol (27 January 1889 – 6 October 1959).

Van der Pol found stable oscillations, now known as **limit cycles**, in electrical circuits employing vacuum tubes. When these circuits are driven near the limit cycle they become entrained, i.e. the driving signal pulls the current along with it.

Figure: An RCA 808 vacuum tube

Van der Pol and his colleague van der Mark reported in **Nature**[1] that at certain drive frequencies an irregular noise was heard. This irregular noise was always heard near the natural entrainment frequencies. This was **one of the first discovered instances of deterministic chaos.**

The van der Pol equation has a long history of being used in both the physical and biological sciences. For instance, in biology, Fitzhugh and Nagumo extended the equation in a planar field as a model for action potentials of neurons. The equation has also been utilized in seismology to model the two plates in a geological fault.

[1] Balth van der Pol and J. van der Mark, *Frequency Demultiplication*, Nature **120**, 363–364 (10 September 1927); doi:10.1038/120363a0

The Van der Pol equation —

$$y'' - \mu(1 - y^2)y' + y = 0,$$

is a model of a non-linear electrical circuit, and the solution has a limit cycle.

- **y** is the position coordinate
- μ is a scalar parameter indicating the strength of the nonlinear damping.

Depending on the damping coefficient μ we get varying behavior:

- When $\mu < 0$, the system will be damped, and $\lim_{t\to\infty} y(t) \to 0$.
- When $\mu = 0$, there is no damping, and we get a simple harmonic oscillator.
- When $\mu \geq 0$, the system will enter a limit cycle, where energy continues to be conserved.

As usual we can transform a higher-order ODE into a system of simultaneous ODEs (let $y_1 = y,\ y_2 = y'$):

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y_2 \\ -y_1 + \mu(1-y_1^2)y_2 \end{bmatrix}.$$

We can also introduce the (standard) transformation

$$\begin{cases} x &= y \\ z &= y' - \mu\left(y - \frac{y^3}{3}\right) \end{cases}$$

and let $F(y) = \mu\left(\frac{y^3}{3} - y\right)$.

Now,

$$x' = y' = \{\text{using the } z\text{-expression}\} = z + \mu\left(x - \frac{x^3}{3}\right)$$

and,

$$\begin{aligned} z' &= y'' - \mu y'\left(1 - y^2\right) \\ &= \underbrace{-\mu(x^2-1)y' - y}_{\text{From Eqn.}} - \mu(1-x^2)y' = -y = -x \end{aligned}$$

This transformation puts the equation into the form:

$$\begin{bmatrix} x' \\ z' \end{bmatrix} = \begin{bmatrix} z - \mu\left(\frac{x^3}{3} - x\right) \\ -x \end{bmatrix},$$

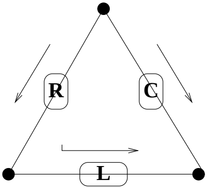
which is a particular case of **Lienard's Equation**

$$\begin{bmatrix} x' \\ z' \end{bmatrix} = \begin{bmatrix} z - f(x) \\ -x \end{bmatrix},$$

with $f(x) = \mu\left(\frac{x^3}{3} - x\right)$.

Taking a Step Back: Where did the Equation Come From???

Consider the a simple circuit with a Resistor (R), a Capacitor (C), and an Inductor (L):



Let i_R, i_L, and i_c be the currents through the resistor, inductor, and capacitor respectively.

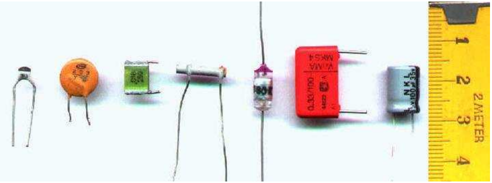
Kirchhoff's Current Law (KCL) says:

$$i_R = i_L = -i_c.$$

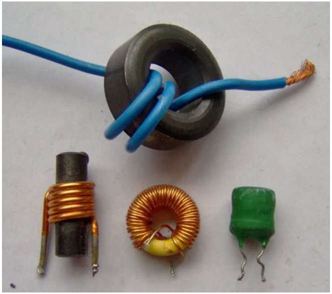
(Current into a node = current out of the node)

Quick Reference: Electronic Components

- **R** — A resistor is a two-terminal electrical or electronic component that resists an electric current by producing a voltage drop between its terminals in accordance with Ohm's law ($R = V/I$). The electrical resistance is equal to the voltage drop across the resistor divided by the current through the resistor.



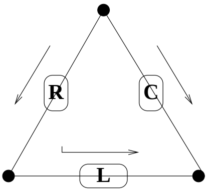
- **C** — A capacitor is an electrical device that can store energy in the electric field between a pair of closely-spaced conductors (called 'plates'). When voltage is applied to the capacitor, electric charges of equal magnitude, but opposite polarity, build up on each plate. Capacitors are used as energy-storage devices. They can also be used to differentiate between high-frequency and low-frequency signals and this makes them useful in electronic filters.



- **L** — Inductance is an effect which results from the magnetic field that forms around a current carrying conductor. Inductance is a measure of the generated electro-magnetic-field for a unit change in current. The inductance of a conductor is increased by **coiling** the conductor such that the magnetic flux encloses all of the coils.

Looking at the RCL circuit

Let α denote the lower left node, γ the lower right node, and β the top node in our circuit:



The **voltage drop** across each branch can be expressed as:

$$v_R = V(\beta) - V(\alpha), \quad v_L = V(\alpha) - V(\gamma), \quad v_c = V(\beta) - V(\gamma).$$

Kirchhoff's Voltage Law (KVL) says:

$$v_R + v_L - v_c = 0.$$

The Resistor branch — Ohm's Law

The relation between the current flowing through a resistor and the voltage drop across the same resistor is governed by Ohms law, $(i_R * R = v_R)$ here we leave it as a general function:

$$f(i_R) = v_R.$$

The Inductor branch — Faraday's Law

The relation between current and voltage in the inductor branch is governed by Faraday's law:

$$L\frac{di_L(t)}{dt} = v_L(t),$$

$L > 0$ is the inductance.

The Capacitor Branch

The relation between current and voltage in the capacitor branch is governed by the following (nameless) law:

$$C\frac{dv_c(t)}{dt} = i_c(t),$$

$C > 0$ is the capacitance.

Collecting the equations...

$$\begin{cases} i_R = i_L = -i_c & \text{(KCL)} \\ v_R + v_L - v_c = 0 & \text{(KVL)} \\ f(i_R) = v_R & \text{(Ohm's Law)} \\ L\frac{di_L(t)}{dt} = v_L(t) & \text{(Faraday's Law)} \\ C\frac{dv_c(t)}{dt} = i_c(t) & \end{cases}$$

For historical reasons, we elect to express our equations in terms of (i_L, v_c):

$$\begin{cases} L\dfrac{di_L(t)}{dt} = v_L = v_c - f(i_L) \\ C\dfrac{dv_c(t)}{dt} = i_c(t) = -i_L(t) \end{cases}$$

Almost there...

We have

$$\begin{cases} L\dfrac{di_L}{dt} = v_c - f(i_L) \\ C\dfrac{dv_c}{dt} = -i_L. \end{cases}$$

By rescaling we can set $L = C = 1$, which with $(x = i_L,\ z = v_c)$ gives us **Lienard's Equation**

$$\begin{cases} x' = z - f(x) \\ z' = -x. \end{cases}$$

In the case $f(x) = R \cdot x$ (Linear Ohm's Law), $(x, z) = (0, 0)$ is an asymptotically stable equilibrium. (Every initial state tends to $(0, 0)$).

Van der Pol's Equation (again)

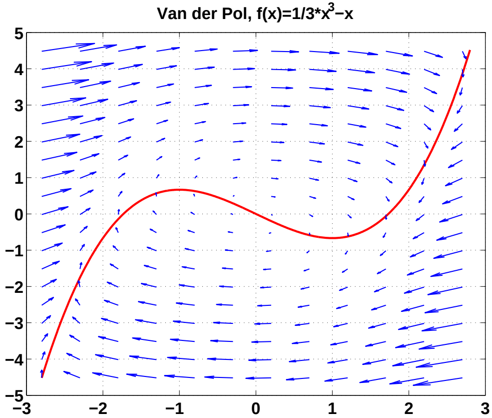
If we have an **active resistor** which follows Ohm's Generalized Law

$$v_R = R\left[\frac{i_R^3}{3} - i_R\right],$$

then $f(x) = \mu\left(\frac{x^3}{3} - x\right)$ in Lienard's Equation ($\mu = R$).

$\Rightarrow$ **Van der Pol's Equation.**

$$\begin{bmatrix} x' \\ z' \end{bmatrix} = \begin{bmatrix} z - \mu\left(\frac{x^3}{3} - x\right) \\ -x \end{bmatrix}$$



Stability of the Origin

Linearizing around the origin gives us:

$$\begin{bmatrix} x' \\ z' \end{bmatrix} = \begin{bmatrix} \mu & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix},$$

with eigenvalues $\lambda_{\pm} = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}$, and eigenvectors

$$\vec{e}_{+} = \begin{bmatrix} \dfrac{-2}{\mu - \sqrt{\mu^2 - 4}} \\ \\ 1 \end{bmatrix}, \quad \vec{e}_{-} = \begin{bmatrix} \dfrac{-2}{\mu + \sqrt{\mu^2 - 4}} \\ \\ 1 \end{bmatrix}.$$

Stability of the Origin: Eigenvalue Structure

μ	$\lambda_{\pm}$	**Comment**
$[-\infty, 0)$	Real($\lambda_{\pm}$)< 0	Origin Stable
0	$\lambda_{\pm} = \pm i$	Marginally Stable/Unstable
$(0, \infty]$	Real($\lambda_{\pm}$)> 0	Origin Unstable
$(0, 2)$	Imag($\lambda_{\pm}$)$\neq 0$	
$[2, \infty]$	Imag($\lambda_{\pm}$)$= 0$	

Also, as $\mu \to \infty$

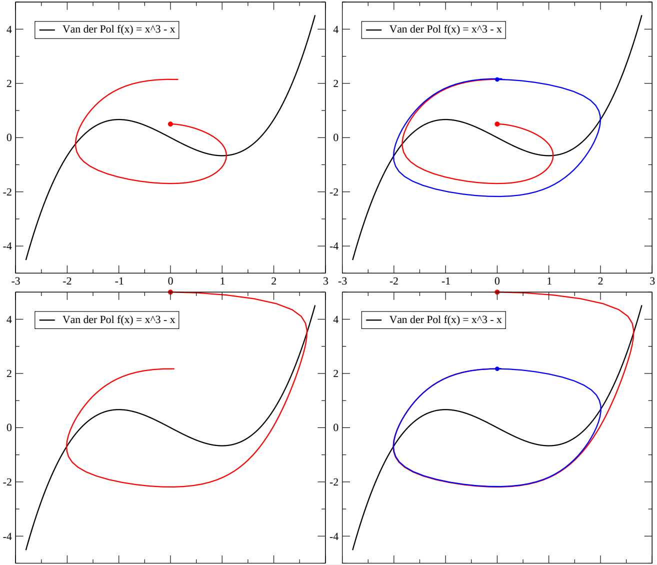
$$\lambda_{+} \sim \mu, \quad \text{and} \quad \lim_{\mu \to \infty} \lambda_{-} \to 0.$$

Leading to more "skew" in the solution...

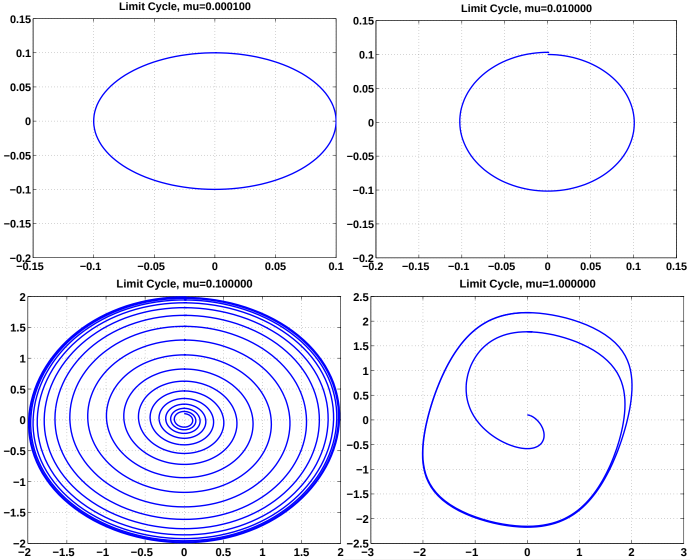
Code Fragments, 9-stage, 7th-order RK

```
f = inline('[y(2) + mu*y(1) - mu*y(1)^3/3; -y(1)]','mu','t','y');
y = [0; 0.1];  ctr=0;
while( go == 1 );
  yn = y(:,ctr+1);
  k1 = f(mu,t, yn);
  k2 = f(mu,t+h/6, yn + h*k1/6);
  k3 = f(mu,t+h/3, yn + h*k2/3);
  k4 = f(mu,t+h/2, yn + h*(k1/8+3*k3/8));
  k5 = f(mu,t+2*h/11, yn + h*(148*k1/1331 + 150*k3/1331 - 56*k4/1331));
  k6 = f(mu,t+2*h/3, yn + h*(-404*k1/243 - 170*k3/27 + 4024*k4/1701 + ...
          10648*k5/1701));
  k7 = f(mu,t+6*h/7, yn + h*(2466*k1/2401 + 1242*k3/343 - ...
          19176*k4/16807 - 51909*k5/16807 + 1053*k6/2401));
  k8 = f(mu,t, yn + h*(5*k1/154+96*k4/539-1815*k5/20384- ...
          405*k6/2464+49*k7/1144));
  k9 = f(mu,t+h, yn + h*(-113*k1/32 - 195*k3/22 + 32*k4/7 ...
          + 29403*k5/3584 -729*k6/512 + 1029*k7/1408 + 21*k8/16));
  ynext = yn + h*(32*k4/105 + 1771561*k5/6289920 + 243*k6/2560 + ...
          16807*k7/74880 + 77*k8/1440 + 11*k9/270);
  y = [y ynext];  ctr = ctr+1;
end
```

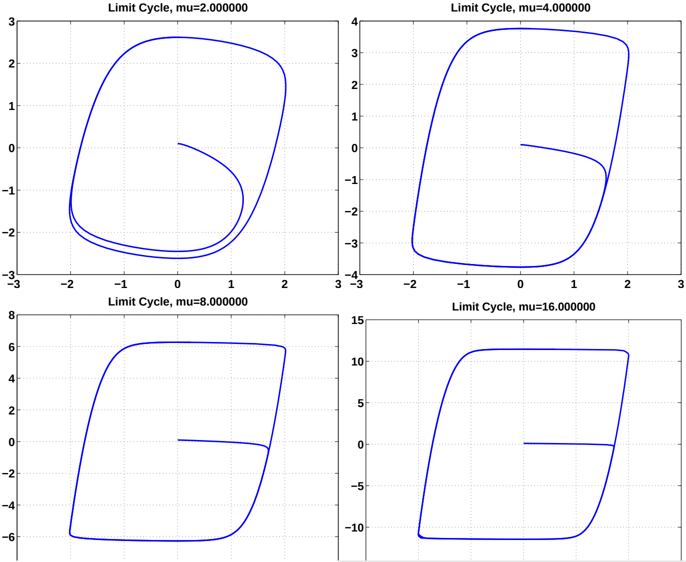
Limit Cycles for $\mu = 1$



Solutions for $\mu \in \{0.0001, 0.01, 0.1, 1\}$

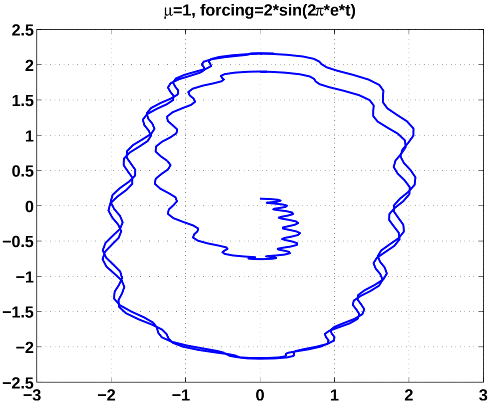


Solutions for $\mu \in \{2, 4, 8, 16\}$



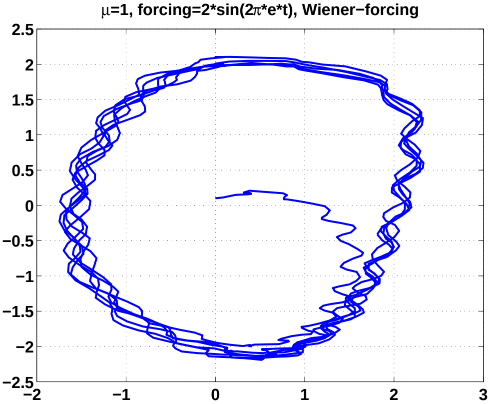
Forced Oscillation

$$y'' - \mu(1 - y^2)y' + y + A\sin(\omega t) = 0, \ [\mu, \ A, \ \omega] = [1, \ 2, \ 2\pi e]$$



Randomly Forced Oscillation

$$y'' - \mu(1-y^2)y' + y + A\sin(\omega t) + W(t) = 0,\ [\mu,\ A,\ \omega] = [0.001,\ 50,\ 2\pi e]$$



Where $W(t)$ is a Wiener process.