Numerical Solutions to Differential Equations

Lecture Notes #21 — Higher Order Equations

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Outline

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   - Numerical Results, Bending with Variable Beam Width

We re-visit the beam-bending problem

$$\frac{d^2}{dx^2}\left[E(x)I(x)\frac{d^2y(x)}{dx^2}\right] = p(x), \quad + \text{BCs}$$

For now, let's assume nothing, *i.e.* $E(x)$ and $I(x)$ are functions.

Differentiating gives

$$\begin{aligned} E(x)I(x)\frac{d^4}{dx^4}\Big[y(x)\Big] &+ 2\frac{d}{dx}\Big[E(x)I(x)\Big]\frac{d^3}{dx^3}\Big[y(x)\Big] \\ &+ \frac{d^2}{dx^2}\Big[E(x)I(x)\Big]\frac{d^2}{dx^2}\Big[y(x)\Big] = p(x). \end{aligned}$$

... and differentiating through the final terms give

$$E(x)I(x)\frac{d^4}{dx^4}\Big[y(x)\Big] + 2\Big[E'(x)I(x) + E(x)I'(x)\Big]\frac{d^3}{dx^3}\Big[y(x)\Big]$$
$$+\Big[E''(x)I(x) + 2E'(x)I'(x) + E(x)I''(x)\Big]\frac{d^2}{dx^2}\Big[y(x)\Big] = p(x).$$

We simplify the problem a bit by assuming that the beam is made from one uniform material, *i.e.* $E(x) = E$; we still allow for a changing beam profile, affecting the area moment of inertia $I(x)$:

$$E \cdot I(x)\frac{d^4}{dx^4}\Big[y(x)\Big] + 2E \cdot I'(x)\frac{d^3}{dx^3}\Big[y(x)\Big]$$
$$+E \cdot I''(x)\frac{d^2}{dx^2}\Big[y(x)\Big] = p(x).$$

In order to exhaust the discussion of boundary conditions, we assume the beam is fixed at the point $x = 0$:

$y(0) = 0$	no deflection	(BC-1)
$y'(0) = 0$	zero slope	(BC-2)

Figure: Something like this?

and we further assume that at $x = L$ the beam is completely free (unsupported):

$y''(L) = 0$	no bending moment	(BC-3)
$y'''(L) = 0$	no shear force	(BC-4)

Second-Order Finite Difference Approximations

We use the **Central Difference Approximations**, with truncation error $\mathcal{O}(h^2)$

$$
\begin{aligned}
y_n'' &\approx [y_{n+1} - 2y_n + y_{n-1}]/h^2 \\
y_n''' &\approx [y_{n+2} - 2y_{n+1} + 2y_{n-1} - y_{n-2}]/2h^3 \\
y_n'''' &\approx [y_{n+2} - 4y_{n+1} + 6y_n - 4y_{n-1} + y_{n-2}]/h^4
\end{aligned}
$$

Since $y_0 = 0$ is specified (BC-1), we need the equations for $n = 1, 2, \ldots, N$ where $x_n = n(L-0)/N$, and $y(x_n) = y_n$.

We use a central difference for (BC-2) and introduce one external (ghost) node x_{-1}:

$$
y'(0) = y_0' = \frac{y_1 - y_{-1}}{2h} = 0, \qquad (\text{BC-2})_{\text{num}}
$$

Pushing Forward

We note that $(\text{BC-2})_{\text{num}}$ gives

$$\mathbf{y_{-1} = y_1}$$

we will use this fact later...

The numerical versions of (BC-3) and (BC-4) are

$$y_N'' = \frac{y_{N-1} - 2y_N + y_{N+1}}{h^2} = 0 \qquad (\text{BC-3})_{\text{num}}$$

$$y_N''' = \frac{-y_{N-2} + 2y_{N-1} - 2y_{N+1} + y_{N+2}}{2h^3} = 0 \qquad (\text{BC-4})_{\text{num}}$$

$(\text{BC-3})_{\text{num}}$ gives $\quad \mathbf{y_{N+1} = 2y_N - y_{N-1}}$

$(\text{BC-4})_{\text{num}}$ gives $\quad \mathbf{y_{N+2} = y_{N-2} - 4y_{N-1} + 4y_N}$

Derivatives of the Area Moment of Inertia

We also need the first and second derivatives of the area moment of inertia $I(x)$ at nodes $n = 1, 2, \ldots, N$:

$$I'_n = \frac{I_{n+1} - I_{n-1}}{2h} \qquad n = 1, 2, \ldots, N-1$$

$$I'_N = \frac{3I_N - 4I_{N-1} + I_{N-2}}{2h} \qquad \text{one-sided}$$

$$I''_n = \frac{I_{n+1} - 2I_n + I_{n-1}}{h^2} \qquad n = 1, 2, \ldots, N-1$$

$$I''_N = \frac{2I_N - 5I_{N-1} + 4I_{N-2} - I_{N-3}}{h^2} \qquad \text{one-sided}$$

Since there are no additional equations for $I(x)$ we are forced to use one-sided differences at the boundaries.

(All the above finite differences are second order.)

Putting it all Together, $n = 2, \ldots, N-2$

The general linear equation at node n is

$$E \cdot I_n \left[\frac{y_{n+2} - 4y_{n+1} + 6y_n - 4y_{n-1} + y_{n-2}}{h^4} \right] +$$
$$+2E \cdot I_n' \left[\frac{y_{n+2} - 2y_{n+1} + 2y_{n-1} - y_{n-2}}{2h^3} \right] +$$
$$+E \cdot I_n'' \left[\frac{y_{n+1} - 2y_n + y_{n-1}}{h^2} \right] = p_n$$

Note that $E \cdot I_n$, $E \cdot I_n'$, $E \cdot I_n''$, and p_n can be pre-computed as they do not depend on the solution y.

At node $n = 1$

$$E \cdot I_1 \left[\frac{y_3 - 4y_2 + 6y_1 - 4\mathbf{y_0} + \mathbf{y_{-1}}}{h^4} \right] +$$

$$+2E \cdot I_1' \left[\frac{y_3 - 2y_2 + 2\mathbf{y_0} - \mathbf{y_{-1}}}{2h^3} \right] +$$

$$+E \cdot I_1'' \left[\frac{y_2 - 2y_1 + \mathbf{y_0}}{h^2} \right] = p_1$$

Now we use $(\text{BC-1})_{\text{num}}$ $y_0 = 0$ and $(\text{BC-2})_{\text{num}}$ $y_{-1} = y_1$:

$$E \cdot I_1 \left[\frac{y_3 - 4y_2 + 7y_1}{h^4} \right] + 2E \cdot I_1' \left[\frac{y_3 - 2y_2 - y_1}{2h^3} \right] +$$

$$+E \cdot I_1'' \left[\frac{y_2 - 2y_1}{h^2} \right] = p_1$$

At node $n = (N-1)$

$$E \cdot I_{N-1}\left[\frac{\mathbf{y_{N+1}} - 4y_N + 6y_{N-1} - 4y_{N-2} + y_{N-3}}{h^4}\right] +$$

$$+2E \cdot I'_{N-1}\left[\frac{\mathbf{y_{N+1}} - 2y_N + 2y_{N-2} - y_{N-3}}{2h^3}\right] +$$

$$+E \cdot I''_{N-1}\left[\frac{y_N - 2y_{N-1} + y_{N-2}}{h^2}\right] = p_{N-1}$$

Now we use (BC-3)'$_{\text{num}}$ $y_{N+1} = 2y_N - y_{N-1}$

$$E \cdot I_{N-1}\left[\frac{\mathbf{2y_N} - \mathbf{y_{N-1}} - 4y_N + 6y_{N-1} - 4y_{N-2} + y_{N-3}}{h^4}\right] +$$

$$+2E \cdot I'_{N-1}\left[\frac{\mathbf{2y_N} - \mathbf{y_{N-1}} - 2y_N + 2y_{N-2} - y_{N-3}}{2h^3}\right] +$$

$$+E \cdot I''_{N-1}\left[\frac{y_N - 2y_{N-1} + y_{N-2}}{h^2}\right] = p_{N-1}$$

At node $n = N$

$$E \cdot I_N \left[\frac{\mathbf{y_{N+2}} - 4\mathbf{y_{N+1}} + 6y_N - 4y_{N-1} + y_{N-2}}{h^4} \right] +$$

$$+2E \cdot I'_N \left[\frac{\mathbf{y_{N+2}} - 2\mathbf{y_{N+1}} + 2y_{N-1} - y_{N-2}}{2h^3} \right] +$$

$$+E \cdot I''_N \left[\frac{\mathbf{y_{N+1}} - 2y_N + y_{N-1}}{h^2} \right] = p_N$$

Now we use

$$\text{(BC-3)'}_{\text{num}} \qquad y_{N+1} = 2y_N - y_{N-1}$$

$$\text{(BC-4)'}_{\text{num}} \qquad y_{N+2} = y_{N-2} - 4y_{N-1} + 4y_N$$

At node $n = N$

$$E \cdot I_N \left[\frac{\mathbf{y_{N-2}} - \mathbf{4y_{N-1}} + \mathbf{4y_N} - 4(\mathbf{2y_N} - \mathbf{y_{N-1}}) + 6y_N - 4y_{N-1} + y_{N-2}}{h^4} \right] +$$

$$+2E \cdot I'_N \left[\frac{\mathbf{y_{N-2}} - \mathbf{4y_{N-1}} + \mathbf{4y_N} - 2(\mathbf{2y_N} - \mathbf{y_{N-1}}) + 2y_{N-1} - y_{N-2}}{2h^3} \right] +$$

$$+E \cdot I''_N \left[\frac{(\mathbf{2y_N} - \mathbf{y_{N-1}}) - 2y_N + y_{N-1}}{h^2} \right] = p_N$$

$$E \cdot I_N \left[\frac{2y_{N-2} - 4y_{N-1} + 2y_N}{h^4} \right] + 2E \cdot I'_N \left[\frac{\mathbf{0}}{2h^3} \right] + E \cdot I''_N \left[\frac{\mathbf{0}}{h^2} \right] = p_N$$

Code: Beam-Bending Segment #1

```
% Beam length
L = 1;

% These boundary conditions are explicitly OR implicitly enforced
% in the equations
a = 0; ya = 0;           y_slope_a = 0;
b = L; y_moment_b = 0; y_shear_b = 0;

% Define the grid
N = 64;
h = (b-a)/(N-1);
x = (a:h:b).';

% Young's Modulus
global E
E = 1;
```

Code: Beam-Bending Segment #2

```
function EI = EI(x)
  global E
  EI = E * (10*ones(size(x))-x/2);
endfunction

function p = p(x)
  p = ones(size(x));
endfunction

EI_value = EI(x);

EI_slope = zeros(size(x));
EI_slope(2:(N-1)) = (EI_value(((2:(N-1))+1))-EI_value(((2:(N-1))-1))) \
    / (2*h);
EI_slope(N) = (3*EI_value(N)-4*EI_value(N-1)+EI_value(N-2))/(2*h);
```

Code: Beam-Bending Segment #3

```
EI_curvature = zeros(size(x));
EI_curvature(2:(N-1)) = ( EI_value(((2:(N-1))+1))  - \
                          2*EI_value(((2:(N-1)))) + EI_value(((2:(N-1))-1)) ) /
EI_curvature(N) = ( 2*EI_value(N) - 5*EI_value(N-1) + 4*EI_value(N-2) \
                    - EI_value(N-3) ) / (h*h);

% Build the linear system
A   = zeros(N,N);
rhs = zeros(N,1);

% node 0 --- Fixed Boundary (y_0 = 0)
A(1,1) = 1;
rhs(1) = ya;
```

Code: Beam-Bending Segment #4

```
% node 1 --- row 2 in the matrix

% coefficients for "y_3"
A(2,4) = (   EI_value(2)/h^4 + 2*EI_slope(2)/(2*h^3) );

% coefficients for "y_2"
A(2,3) = (-4*EI_value(2)/h^4 + \
          -4*EI_slope(2)/(2*h^3) + EI_curvature(2)/h^2 );

% coefficients for "y_1"
A(2,2) = ( 7*EI_value(2)/h^4 - 2*EI_slope(2)/(2*h^3) + \
          -2*EI_curvature(2)/h^2 );

rhs(2) = p(x(2));
```

Code: Beam-Bending Segment #5

```
% nodes 2 to N-2 --- rows 3 to N-2 in the matrix
for k = 3:(N-2)
  % coefficients for y_{n+2}
  A(k,k+2) = EI_value(k)/h^4 + 2*EI_slope(k)/(2*h^3);
  % coefficients for y_{n+1}
  A(k,k+1) = -4*EI_value(k)/h^4 -4*EI_slope(k)/(2*h^3) + \
      EI_curvature(k)/h^2;
  % coefficients for y_{n}
  A(k,k) = 6*EI_value(k)/h^4 -2*EI_curvature(k)/h^2;
  % coefficients for y_{n-1}
  A(k,k-1) = -4*EI_value(k)/h^4 +4*EI_slope(k)/(2*h^3) + \
      EI_curvature(k)/h^2;
  % coefficients for y_{n-2}
  A(k,k-2) = EI_value(k)/h^4 - 2*EI_slope(k)/(2*h^3);
  % right-hand-side
  rhs(k) = p(x(k));
end
```

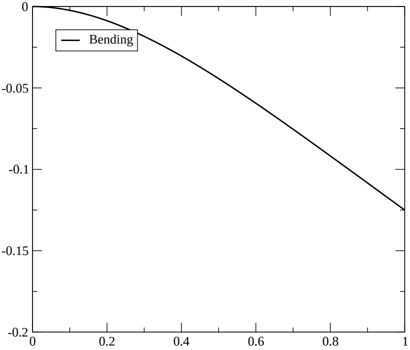
Code: Beam-Bending Segment #6

```
% node N-1

% y_{N}
A(N-1,N) = -2*EI_value(N-1)/h^4 + EI_curvature(N-1)/h^2;

% y_{N-1}
A(N-1,N-1) = 5*EI_value(N-1)/h^4 + -2*EI_slope(N-1)/(2*h^3) + \
    -2*EI_curvature(N-1)/h^2;

% y_{N-2}
A(N-1,N-2) = -4*EI_value(N-1)/h^4 + 4*EI_slope(N-1)/(2*h^3) + \
    EI_curvature(N-1)/h^2;

% y_{N-3}
A(N-1,N-3) = EI_value(N-1)/h^4 - 2*EI_slope(N-1)/(2*h^3);

% right-hand-side
rhs(N-1) = p(x(N-1));
```

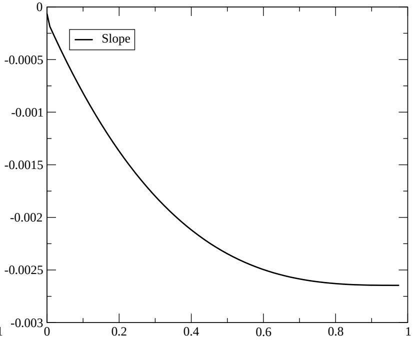
Code: Beam-Bending Segment #7

```
% node N

% y_{N}
A(N,N) = 2*EI_value(N)/h^4;
% y_{N-1}
A(N,N-1) = -4*EI_value(N)/h^4;

% y_{N-2}
A(N,N-2) = 2*EI_value(N)/h^4;

% right-hand-side
rhs(N) = p(x(N));

% solve for the deflection
y = A\rhs;
```

Numerical Results (Uniform Load / Beam, $I(x) = 1$)

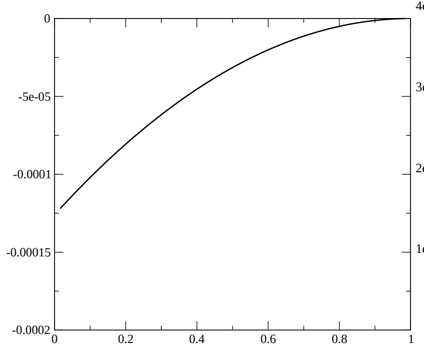


Bending

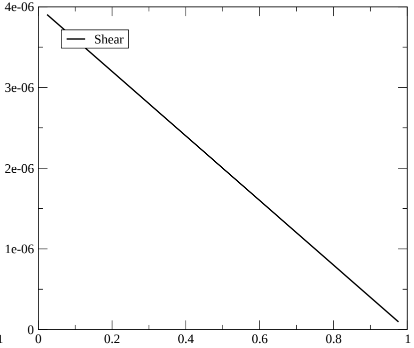


Slope

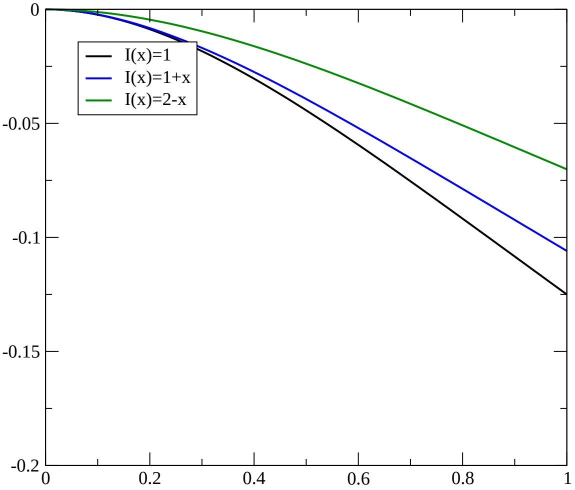
Numerical Results (Uniform Load / Beam, $I(x) = 1$)



Moment



Shear


0
-0.05
-0.1
-0.15
-0.2
0
0.2
0.4
0.6
0.8
1
I(x)=1
I(x)=1+x
I(x)=2-x

Classical Beam Theory

- Simplification of the linear theory of elasticity; means of calculating the load-carrying and deflection characteristics of beams.
- Covers the case for small deflections, subject to lateral loads.
- History back to $\sim$1750.
- Early applications: development of the Eiffel Tower and the Ferris wheel in the late 19th century.
- Cornerstone of engineering and an enabler of the Second Industrial Revolution.
    - The Second Industrial Revolution was characterized by the build out of railroads, large scale iron and steel production, widespread use of machinery in manufacturing, greatly increased use of steam power, use of oil, beginning of electricity and by electrical communications.