The Van der Pol Oscillator

Return to Physics — Circuit Analysis

The Van der Pol Oscillator, again...

Numerical Solutions to Differential Equations

Lecture Notes #13
The Van der Pol Oscillator

Peter Blomgren,
⟨blomgren.peter@gmail.com⟩

Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

http://terminus.sdsu.edu/

Spring 2015
Outline

1. The Van der Pol Oscillator
   - Second order ODE $\leadsto$ 2D system
   - 2D-system $\leadsto$ Lienard Equation

2. Return to Physics — Circuit Analysis
   - R-C-L Circuit

3. The Van der Pol Oscillator, again...
   - (Physical) Stability Analysis of the Origin
   - Finally, Computations

Peter Blomgren, ⟨blomgren.peter@gmail.com⟩
The van der Pol oscillator was originally “discovered” by the Dutch electrical engineer and physicist Balthasar van der Pol (27 January 1889 – 6 October 1959).

Van der Pol found stable oscillations, now known as **limit cycles**, in electrical circuits employing vacuum tubes. When these circuits are driven near the limit cycle they become entrained, i.e. the driving signal pulls the current along with it.

**Figure:** An RCA 808 vacuum tube
Van der Pol and his colleague van der Mark reported in Nature¹ that at certain drive frequencies an irregular noise was heard. This irregular noise was always heard near the natural entrainment frequencies. This was one of the first discovered instances of deterministic chaos.

The van der Pol equation has a long history of being used in both the physical and biological sciences. For instance, in biology, Fitzhugh and Nagumo extended the equation in a planar field as a model for action potentials of neurons. The equation has also been utilized in seismology to model the two plates in a geological fault.

¹Balth van der Pol and J. van der Mark, Frequency Demultiplication, Nature 120, 363–364 (10 September 1927); doi:10.1038/120363a0
The Van der Pol equation —

\[ y'' - \mu (1 - y^2)y' + y = 0, \]

is a model of a non-linear electrical circuit, and the solution has a limit cycle.

- \( y \) is the position coordinate
- \( \mu \) is a scalar parameter indicating the strength of the nonlinear damping.
\[ y'' - \mu(1 - y^2)y' + y = 0 \]

Depending on the damping coefficient \( \mu \) we get varying behavior:

- When \( \mu < 0 \), the system will be damped, and \( \lim_{t \to \infty} y(t) \to 0 \).
- When \( \mu = 0 \), there is no damping, and we get a simple harmonic oscillator.
- When \( \mu \geq 0 \), the system will enter a limit cycle, where energy continues to be conserved.

As usual we can transform a higher-order ODE into a system of simultaneous ODEs (let \( y_1 = y \), \( y_2 = y' \)):

\[
\begin{bmatrix}
  y_1' \\
  y_2'
\end{bmatrix}
= \begin{bmatrix}
  y_2 \\
  -y_1 + \mu(1 - y_1^2)y_2
\end{bmatrix}.
\]
We can also introduce the (standard) transformation

\[
\begin{cases}
    x &= y \\
    z &= y' - \mu \left( y - \frac{y^3}{3} \right)
\end{cases}
\]

and let \( F(y) = \mu \left( \frac{y^3}{3} - y \right) \).

Now,

\[
x' = y' = \{ \text{using the } z\text{-expression} \} = z + \mu \left( x - \frac{x^3}{3} \right)
\]

and,

\[
z' = y''' - \mu y' \left( 1 - y^2 \right)
\]

\[
= -\mu(x^2 - 1)y' - y - \mu(1 - x^2)y' = -y = -x
\]

From Eqn.
This transformation puts the equation into the form:

\[
\begin{bmatrix}
  x' \\
  z'
\end{bmatrix} = \begin{bmatrix}
  z - \mu \left( \frac{x^3}{3} - x \right) \\
  -x
\end{bmatrix},
\]

which is a particular case of **Lienard’s Equation**

\[
\begin{bmatrix}
  x' \\
  z'
\end{bmatrix} = \begin{bmatrix}
  z - f(x) \\
  -x
\end{bmatrix},
\]

with \( f(x) = \mu \left( \frac{x^3}{3} - x \right) \).
Consider the a simple circuit with a Resistor (R), a Capacitor (C), and an Inductor (L):

Let $i_R$, $i_L$, and $i_c$ be the currents through the resistor, inductor, and capacitor respectively.

**Kirchhoff’s Current Law** (KCL) says:

$$i_R = i_L = -i_c.$$  

(Current into a node = current out of the node)
**R** — A resistor is a two-terminal electrical or electronic component that resists an electric current by producing a voltage drop between its terminals in accordance with Ohm’s law \((R = V/I)\). The electrical resistance is equal to the voltage drop across the resistor divided by the current through the resistor.
A capacitor is an electrical device that can store energy in the electric field between a pair of closely-spaced conductors (called 'plates'). When voltage is applied to the capacitor, electric charges of equal magnitude, but opposite polarity, build up on each plate. Capacitors are used as energy-storage devices. They can also be used to differentiate between high-frequency and low-frequency signals and this makes them useful in electronic filters.
L — Inductance is an effect which results from the magnetic field that forms around a current carrying conductor. Inductance is a measure of the generated electro-magnetic-field for a unit change in current. The inductance of a conductor is increased by **coiling** the conductor such that the magnetic flux encloses all of the coils.
Looking at the RCL circuit

Let $\alpha$ denote the lower left node, $\gamma$ the lower right node, and $\beta$ the top node in our circuit:

![RCL Circuit Diagram]

The **voltage drop** across each branch can be expressed as:

$$v_R = V(\beta) - V(\alpha), \quad v_L = V(\alpha) - V(\gamma), \quad v_c = V(\beta) - V(\gamma).$$

**Kirchhoff’s Voltage Law** (KVL) says:

$$v_R + v_L - v_c = 0.$$
Ohm’s and Faraday’s Laws

**The Resistor branch — Ohm’s Law**
The relation between the current flowing through a resistor and the voltage drop across the same resistor is governed by Ohms law, 
\( i_R \times R = v_R \) here we leave it as a general function:

\[ f(i_R) = v_R. \]

**The Inductor branch — Faraday’s Law**
The relation between current and voltage in the inductor branch is governed by Faraday’s law:

\[ L \frac{di_L(t)}{dt} = v_L(t), \]

\( L > 0 \) is the inductance.
The Capacitor Branch
The relation between current and voltage in the capacitor branch is governed by the following (nameless) law:

\[ C \frac{d v_c(t)}{d t} = i_c(t), \]

\( C > 0 \) is the capacitance.
Collecting the equations...

\[
\begin{align*}
    i_R &= i_L = -i_c \quad \text{(KCL)} \\
    v_R + v_L - v_c &= 0 \quad \text{(KVL)} \\
    f(i_R) &= v_R \quad \text{(Ohm’s Law)} \\
    L \frac{di_L(t)}{dt} &= v_L(t) \quad \text{(Faraday’s Law)} \\
    C \frac{dv_c(t)}{dt} &= i_c(t)
\end{align*}
\]

For historical reasons, we elect to express our equations in terms of \((i_L, v_c)\):  

\[
\begin{align*}
    L \frac{di_L(t)}{dt} &= v_L = v_c - f(i_L) \\
    C \frac{dv_c(t)}{dt} &= i_c(t) = -i_L(t)
\end{align*}
\]
Almost there...

We have

\[
\begin{align*}
L \frac{di_L}{dt} &= v_c - f(i_L) \\
C \frac{dv_c}{dt} &= -i_L.
\end{align*}
\]

By rescaling we can set \( L = C = 1 \), which with \((x = i_L, z = v_c)\) gives us **Lienard’s Equation**

\[
\begin{align*}
x' &= z - f(x) \\
z' &= -x.
\end{align*}
\]

In the case \( f(x) = R \cdot x \) (Linear Ohm’s Law), \((x, z) = (0, 0)\) is an asymptotically stable equilibrium. (Every initial state tends to \((0, 0)\)).
If we have an **active resistor** which follows Ohm’s Generalized Law

\[ v_R = R \left( \frac{i_R^3}{3} - i_R \right), \]

then \( f(x) = \mu \left( \frac{x^3}{3} - x \right) \) in Lienard’s Equation (\( \mu = R \)).

⇒ **Van der Pol’s Equation.**
Van der Pol's Equation

\[
\begin{bmatrix}
  x' \\
  z'
\end{bmatrix} = \begin{bmatrix}
  z - \mu \left( \frac{x^3}{3} - x \right) \\
  -x
\end{bmatrix}
\]

Van der Pol, \( f(x) = \frac{1}{3} x^3 - x \)
Stability of the Origin

Linearizing around the origin gives us:

\[
\begin{bmatrix}
    x' \\
    z'
\end{bmatrix} =
\begin{bmatrix}
    \mu & 1 \\
    -1 & 0
\end{bmatrix}
\begin{bmatrix}
    x \\
    z
\end{bmatrix},
\]

with eigenvalues \( \lambda_\pm = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2} \), and eigenvectors

\[
\tilde{e}_+ = \begin{bmatrix}
    -2 \\
    \mu - \sqrt{\mu^2 - 4} \\
    1
\end{bmatrix}, \quad \tilde{e}_- = \begin{bmatrix}
    -2 \\
    \mu + \sqrt{\mu^2 - 4} \\
    1
\end{bmatrix}.
\]
Stability of the Origin: Eigenvalue Structure

<table>
<thead>
<tr>
<th>$\mu$</th>
<th>$\lambda_{\pm}$</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[-\infty, 0)$</td>
<td>Real($\lambda_{\pm}$) &lt; 0</td>
<td>Origin Stable</td>
</tr>
<tr>
<td>0</td>
<td>$\lambda_{\pm} = \pm i$</td>
<td>Marginally Stable/Unstable</td>
</tr>
<tr>
<td>$(0, \infty]$</td>
<td>Real($\lambda_{\pm}$) &gt; 0</td>
<td>Origin Unstable</td>
</tr>
<tr>
<td>(0, 2)</td>
<td>Imag($\lambda_{\pm}$) $\neq 0$</td>
<td></td>
</tr>
<tr>
<td>[2, $\infty]$</td>
<td>Imag($\lambda_{\pm}$) = 0</td>
<td></td>
</tr>
</tbody>
</table>

Also, as $\mu \to \infty$

$$\lambda_+ \sim \mu, \quad \text{and} \quad \lim_{\mu \to \infty} \lambda_- \to 0.$$ 

Leading to more “skew” in the solution...
Code Fragments, 9-stage, 7th-order RK

```matlab
f = inline('[y(2) + mu*y(1) - mu*y(1)ˆ3/3; -y(1)]', 'mu', 't', 'y');
y = [0; 0.1]; ctr = 0;
while( go == 1 );
    yn = y(:,ctr+1);
    k1 = f(mu,t, yn);
    k2 = f(mu,t+h/6, yn + h*k1/6);
    k3 = f(mu,t+h/3, yn + h*k2/3);
    k4 = f(mu,t+h/2, yn + h*(k1/8+3*k3/8));
    k5 = f(mu,t+2*h/11, yn + h*(148*k1/1331 + 150*k3/1331 - 56*k4/1331));
    k6 = f(mu,t+2*h/3, yn + h*(-404*k1/243 - 170*k3/27 + 4024*k4/1701 + ...
     10648*k5/1701));
    k7 = f(mu,t+6*h/7, yn + h*(2466*k1/2401 + 1242*k3/343 - ...
     19176*k4/16807 - 51909*k5/16807 + 1053*k6/2401));
    k8 = f(mu,t, yn + h*(5*k1/154+96*k4/539-1815*k5/20384- ...
     405*k6/2464+49*k7/1144));
    k9 = f(mu,t+h, yn + h*(-113*k1/32 - 195*k3/22 + 32*k4/7 ...  
     + 29403*k5/3584 -729*k6/512 + 1029*k7/1408 + 21*k8/16));
    ynext = yn + h*(32*k4/105 + 1771561*k5/6289920 + 243*k6/2560 + ...
     16807*k7/74880 + 77*k8/1440 + 11*k9/270);
    y = [y ynext]; ctr = ctr+1;
end
```

Peter Blomgren, ⟨blomgren.peter@gmail.com⟩
Limit Cycles for $\mu = 1$
Solutions for $\mu \in \{0.0001, 0.01, 0.1, 1\}$
Solutions for $\mu \in \{2, 4, 8, 16\}$

The Van der Pol Oscillator
Forced Oscillation

\[ y'' - \mu(1 - y^2)y' + y + A \sin(\omega t) = 0, \quad [\mu, A, \omega] = [1, 2, 2\pi e] \]
Randomly Forced Oscillation

\[ y'' - \mu (1 - y^2) y' + y + A \sin(\omega t) + W(t) = 0, \quad [\mu, A, \omega] = [0.001, 50, 2\pi e] \]

Where \( W(t) \) is a Wiener process.

Peter Blomgren, \langle blomgren.peter@gmail.com \rangle