Numerical Solutions to Differential Equations

Lecture Notes

The Finite Element Method #1 — Introduction

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Outline

Introduction to the Finite Element Method (FEM)

Concept: Any solution domain can be divided into sub-domains *(finite elements)*. By assuming a simple form solution in each finite element, the approximate solution of the problem in the entire domain is determined.

FEM has all the advantages of the finite difference approach, and can very easily handle irregular boundaries and non-uniform grids.

FEM is the second approach (of three) to the numerical solution of PDEs developed in the mid-to-late 1900s, roughly:

1950s – 1960s: Finite difference methods
1960s – 1970s: **Finite element methods**
1970s – 1980s: Spectral methods

Prior to jumping into the details, lets look at some applications...

FEM: References

These lectures based on:

[J1987] Claes Johnson, *Numerical Solution of Partial Differential Equations by the Finite Element Method*, Cambridge University Press, 1987. OUT OF PRINT

Revised as

[EEHJ1996] K. Eriksson, D. Estep, P. Hansbo, and C. Johnson, *Computational Differential Equations*, Cambridge University Press, 1996.

"Classical" References:

[C2002] Philippe G. Ciarlet, *The Finite Element Method for Elliptic Problems*, Society for Industrial and Applied Mathematics, 2002.

[ZT2005A] O.C. Zienkiewicz and R.L. Taylor, *The Finite Element Method, 1: The Basis,* 6th Edition, Butterworth and Heneimann, 2005.

[ZT2005B] O.C. Zienkiewicz and R.L. Taylor, *The Finite Element Method, 2: Solid Mechanics,* 6th Edition, Butterworth and Heneimann, 2005.

[ZT2005C] O.C. Zienkiewicz and R.L. Taylor, *The Finite Element Method, 3: Fluid Dynamics,* 6th Edition, Butterworth and Heneimann, 2005.

Finite Element Grid: Geysers Coring Project

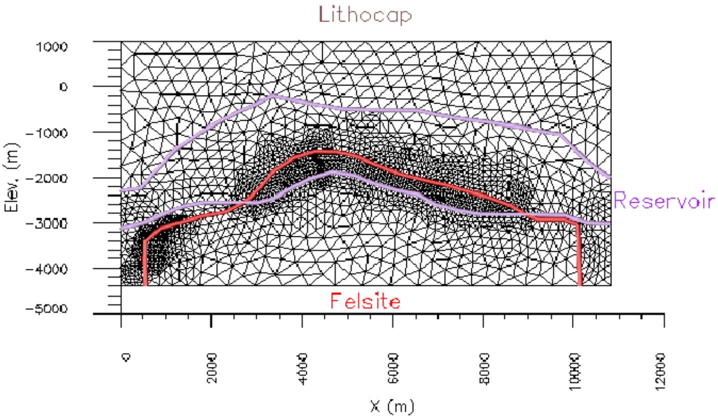


Figure: Cross-sectional geometry used in finite element modeling. Elements shown in black, grid contains 1073 quadratic triangular elements, 2188 nodes. Boundary of permeable reservoir shown in blue, overlapping top boundary of felsite (fine grained volcanic rock) shown in red.

http://www.utdallas.edu/∼brikowi/Publications/Geysers/GRC99_Talk/node19.html

Finite Element Solution: Geysers Coring Project

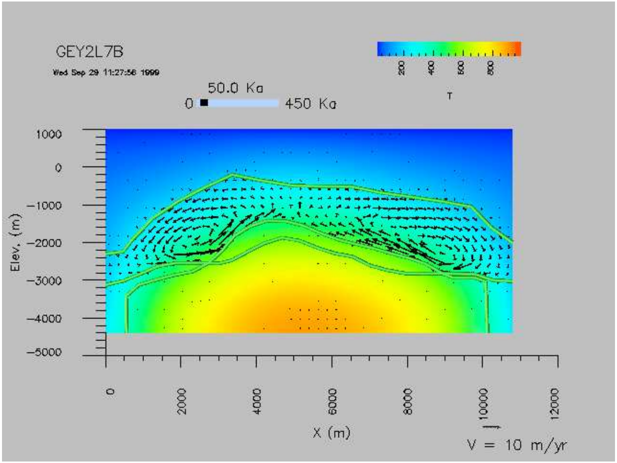


Figure: Temperature (shading) and flow (black vectors, max V=8 m/yr) fields at 50 Ka for mariah model results, SW-NE cross-section, The Geysers. Lines show margins of present-day steam reservoir.

Finite Element Grid: the Adriatic Sea

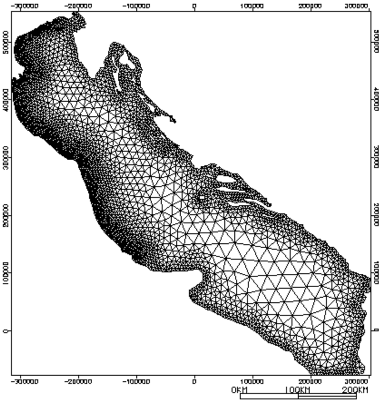


Figure: The discretization of the Adriatic Sea with finite elements. This grid is used in a primitive equation model to study the circulation in the North Adriatic Sea forced by tides and winds.

The complete mesh consists of 3924 nodes and 7034 elements. The resolution of the grid is higher in the coastal areas, especially close to the northern Italian coast.

http://www.isdgm.ve.cnr.it/∼georg/adria/mast36/ad_grid.html

Finite Element Grid: the Venice Lagoon

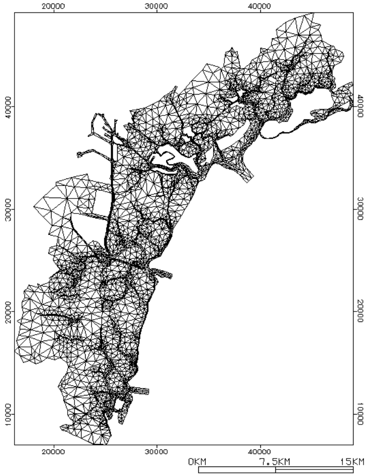


Figure: The grid for the Venice Lagoon contains 4237 nodes and 7666 elements. It can be seen from the figure that all main channels are reproduced. They can be distinguished in the plot by their representation through smaller elements. Bigger elements have been used for the shallower parts where hydrodynamic activity is reduced.

http://www.isdgm.ve.cnr.it/∼georg/venice/model/model.html

The Finite Element Method — Basic Steps

We reformulate the **differential equation**

$$(D) \qquad Lu = f,$$

as an equivalent **variational problem**

$$(V) \qquad \text{Find } u \in V \text{ such that } a(u, v) = (f, v) \text{ for all } v \in V.$$

The solution to this problem can be found from the equivalent **minimization problem**

$$(M) \qquad \text{Find } u \in V \text{ such that } F(u) \leq F(v) \text{ for all } v \in V.$$

We have a lot of work to do... We need to define and understand what this means...

Some Definitions

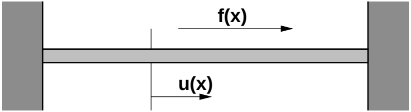
V is a set of admissible functions, *e.g.* the set of piecewise linear functions. [FUNCTION SPACE]

F is a functional which assigns a real value to every function in V, $F : V \to \mathbb{R}$.

$a(\circ, \circ)$ and $(\circ, \circ)$ are inner products defined for the functions in V. — The functional F is defined in terms of these inner products.

Let's consider three one-dimensional examples...

Example #1: Elastic Bar subject to Tangential Load



Consider an elastic bar fixed at both ends, subject to tangential load of intensity $f(x)$. Let $\sigma(x)$ be the traction, and $u(x)$ the tangential displacement under the load.

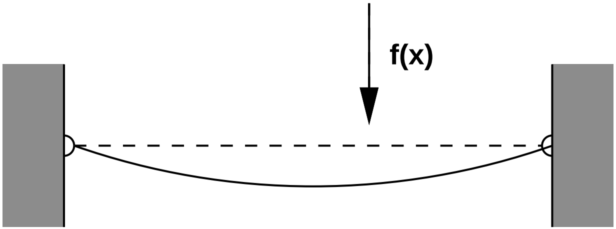
Under the assumption of small displacements and a linearly elastic material, we have

$$\begin{array}{rclr} \sigma & = & Eu' & \text{(Hooke's law)} \\ -\sigma' & = & f & \text{(Equilibrium equation)} \\ u(0) & = & u(1) = 0 & \text{(Boundary conditions)} \end{array}$$

This gives

$$-u''(x) = \frac{1}{E} f(x), \quad u(0) = u(1) = 0.$$

Example #2: Elastic Cord subject to Transverse Load



Consider an elastic cord with tension 1, fixed at both ends and subject to transverse load of intensity $f(x)$. Assuming small displacements, the equation for the transverse displacement is

$$-u''(x) = f(x), \quad u(0) = u(1) = 0.$$

Example #3: Heat Conduction in a Rod

Let u be the temperature and q the heat flow in a conducting rod, subject to a distributed heat source $f(x)$. Assuming zero temperature at the ends (cooling), the stationary heat distribution in the rod is given by

$$\begin{aligned} -q &= ku' && \text{(Fourier's law)} \\ q' &= f && \text{(Conservation of energy)} \\ u(0) &= u(1) = 0 && \text{(BCs)} \end{aligned}$$

where k is the heat conductivity. If k is constant in the rod, then the equation for the heat distribution is

$$-u''(x) = \frac{1}{k} f(x), \quad u(0) = u(1) = 0.$$

Going Forward

We have three examples which are described by the equation

$$(D) \qquad -u''(x) = f(x), \quad u(0) = u(1) = 0.$$

Therefore we don't feel completely silly looking at this problem...

Let V be the space of piecewise continuous functions on $[0, 1]$:

$$V = \left\{ \begin{array}{ll} v: & v \in C[0,1] \\ & v' \text{ piecewise continuous and bounded} \\ & v(0) = v(1) = 0 \end{array} \right\}$$

We define the inner product(s) — [Yeah, they're the same, for now]

$$a(v, w) = \int_0^1 v(x)w(x)\,dx, \quad (v, w) = \int_0^1 v(x)w(x)\,dx.$$

Variational and Minimization Formulation

Further, we define the linear functional $F : V \to \mathbb{R}$:

$$F(v) = \frac{1}{2}a(v', v') - (f, v)$$

The corresponding **minimization problem** is

$$(M) \qquad \text{Find } u \in V \text{ such that } F(u) \leq F(v) \text{ for all } v \in V.$$

The corresponding **variational problem** is

$$(V) \qquad \text{Find } u \in V \text{ such that } a(u', v') = (f, v) \text{ for all } v \in V.$$

Physical Interpretations

In examples #1 and #2, $F(v)$ represents the **total potential energy** associated with the displacement v:

— $a(v', v')$ represents the internal elastic energy, and

— (f, v) the load potential.

(M) corresponds to the fundamental **Principle of minimum potential energy** in mechanics.

[http://en.wikipedia.org/wiki/Minimum_total_potential_energy_principle]

(V) corresponds to the **Principle of virtual work**.

[http://en.wikipedia.org/wiki/Virtual_work]

Showing $(D) \Leftrightarrow (V) \Leftrightarrow (M)$

We have boldly stated that the three problems are equivalent... We need to show it! (In the process, we reveal under what conditions it is true...)

First we show $(D) \Rightarrow (V)$:

Multiply the equation $-u'' = f$ by any function $v \in V$ (a so-called **test-function**), and integrate over $[0, 1]$:

$$-a(u'', v) = (f, v).$$

We use integration by parts on the left-hand-side

$$\begin{aligned} -a(u'', v) &= \int_0^1 u'' v \, dx = -u'(1) \underbrace{v(1)}_{0} + u'(0) \underbrace{v(0)}_{0} + \int_0^1 u' v' \, dx \\ &= \int_0^1 u' v' \, dx = a(u', v'). \end{aligned}$$

Showing $(D) \Leftrightarrow (V) \Leftrightarrow (M)$

We now have

$$a(u', v') = (f, v), \quad \forall v \in V,$$

hence $(D) \Rightarrow (V)$.

Next we show $(V) \Rightarrow (D)$. Assume u satisfies the relation above (V), *i.e.*

$$\int_0^1 u'v'\,dx - \int_0^1 fv\,dx = 0, \quad \forall v \in V.$$

If we make the **additional assumption** that u'' exists and is continuous, then we can integrate by parts in the first term and get

$$-\int_0^1 (u'' + f)v\,dx = 0, \quad \forall v \in V.$$

Since $(u'' + f)$ is continuous this implies $u'' + f = 0$, $x \in (0, 1)$. Hence $(V) \Rightarrow (D)$.

Showing $(D) \Leftrightarrow (V) \Leftrightarrow (M)$

Finally, we show that $(V) \Leftrightarrow (M)$:

First, suppose u is a solution to (V), let $v \in V$ and set $w = v - u$ $(v = u + w$ and $w \in V.)$ We have

$$\begin{aligned} F(v) &= F(u+w) = \tfrac{1}{2}a(u'+w', u'+w') - (f, u+w) \\ &= \underbrace{\frac{1}{2}a(u',u') - (f,u)}_{F(u)} + \underbrace{a(u',w') - (f,w)}_{=0} + \tfrac{1}{2}\underbrace{a(w',w')}_{\geq 0} \\ &\geq F(u) \end{aligned}$$

Hence $(V) \Rightarrow (M)$.

Showing $(D) \Leftrightarrow (V) \Leftrightarrow (M)$

On the other hand, if u is a solution of (M) then for any $v \in V$ and real number $\epsilon \in \mathbb{R}$

$$F(u) \leq F(u + \epsilon v),$$

since $u + \epsilon v \in V$. Thus, the differentiable function

$$g(\epsilon) = F(u+\epsilon v) = \frac{1}{2} a(u', u') + \epsilon a(u', v') + \frac{\epsilon^2}{2} a(v', v') - (f, u) - \epsilon (f, v)$$

has a minimum at $\epsilon = 0$, and hence $g'(0) = 0$. But

$$g'(0) = a(u', v') - (f, v)$$

and we see that u is a solution of (V).

One Hole to Plug: Uniqueness of solutions to (V)

Suppose u_1 and u_2 are solutions of (V), *i.e.* $u_1,\ u_2 \in V$, and

$$\begin{aligned} a(u_1', v') &= (f, v) \qquad \forall v \in V \\ a(u_2', v') &= (f, v) \qquad \forall v \in V \end{aligned}$$

Subtracting these equations, and **choosing** $v = u_1 - u_2 \in V$, we get

$$\int_0^1 (u_1' - u_2')^2 \, dx = 0$$

Which shows that

$$\frac{d}{dx}\left[u_1 - u_2\right](x) = 0, \quad x \in (0, 1)$$

Hence $\left[u_1 - u_2\right](x)$ is constant. Since both $u_1(0) = u_2(0) = 0$ the constant is **0**. This shows uniqueness. $\square$

Moving to a **Finite** Model

With the basic theory out of the way, we are ready to approach the Finite Element Method.

We shall construct a **finite**-dimensional subspace V_h of V (containing piecewise linear functions) as defined earlier.

Let

$$x_n = \sum_{k=1}^{n} h_k, \quad n = 0, 1, \ldots, (N+1), \quad \text{where} \sum_{k-1}^{N+1} h_k = 1, \quad h_k > 0.$$

Note that the points do not have to be equally spaced. Let

$$I_n = [x_{n-1}, x_n],$$

denote the sub-intervals.

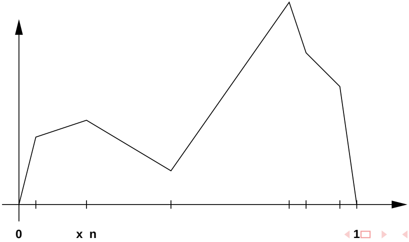
Moving to a **Finite** Model

Let the largest interval

$$h = \max_{n=1,\dots,(N+1)} h_n$$

be a measure of how fine the grid is.

The Finite-Dimensional Subspace: We let V_h be the set of functions v so that v is linear on each sub-interval I_n, continuous on $[0, 1]$ and $v(0) = v(1) = 0$.



FEM with Piecewise Linear Functions

We observe that $V_h \subset V$, *i.e.* all functions in V_h are also members of V.

We parameterize V_h by the values at the nodes

$$\eta_n = v(x_n).$$

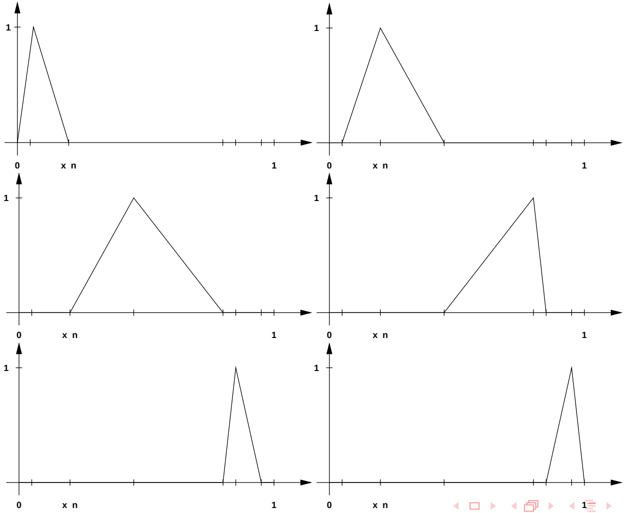
We introduce the **basis functions** $\phi_n \in V_h$ defined by

$$\phi_n(x_m) = \delta_{n,m},$$

i.e. ϕ_n is the continuous piecewise linear functions that takes the value 1 at node point x_n and the value 0 at all other node points.

Basis Functions: "The Tent Functions"

The basis functions for the partition are:



Representing Functions $v \in V_h$

A function $v \in V_h$ has the representation

$$v(x) = \sum_{i=1}^{N} \eta_i \phi_i(x), \quad x \in [0, 1]$$

where $\eta_i = v(x_i)$.

Each $v \in V_h$ can be written as a **unique linear combination** of the basis functions ϕ_i.

V_h is a **linear space of dimension** N with **basis** $\{\phi_i\}_{i=1}^{N}$.

We now, **finally** have the tools and language to formulate the finite element method for the boundary value problem (D)...

Formulating the Finite Element Method

Galerkin's Method/Formulation:

$$(V_h) \qquad \text{Find } u_h \in V_h \text{ so that } a(u_h', v') = (f, v), \; \forall v \in V_h.$$

Ritz' Method/Formulation:

$$(M_h) \qquad \text{Find } u_h \in V_h \text{ such that } F(u_h) \leq F(v), \; \forall v \in V_h.$$

We have defined the basic language and framework for the finite element method. Next time we will start talking about how to find the solutions...