Numerical Solutions to Differential Equations

Lecture Notes — The Finite Element Method #2

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Outline

Quick Recap

Last time we formulated the Galerkin (variational, (V_h)) and Ritz (minimization, (M_h)) methods which will give us the finite element solutions to the differential equation (D):

$$-u'' = f, \quad + \quad \text{Dirichlet Boundary Conditions}$$

The Function Space V_h: Given the basis functions ϕ_i (the "tent" functions) we can write

$$V_h = \left\{ v : \; v = \sum_{i=1}^{N} \eta_i \phi_i(x) \right\}.$$

Quick Recap II/III

Inner Products: We defined the two (for now identical) inner products:

$$a(u,v) = \int_{\mathcal{I}} uv\, dx, \quad (u,v) = \int_{\mathcal{I}} uv\, dx.$$

Further, we defined the energy **functional**:

$$F(u) = \frac{1}{2} a(v', v') - (f, v).$$

Given these definitions we formulated the Galerkin and Ritz problems...

Quick Recap III/III

Galerkin's Method (Variational Approach)

$$(V_h) \qquad \text{Find } u_h \in V_h \text{ so that } a(u_h', v_h') = (f, v_h) \quad \forall v_h \in V_h.$$

Ritz' Method (Minimization Approach)

$$(M_h) \qquad \text{Find } u_h \in V_h \text{ so that } F(u_h) \leq F(v_h) \quad \forall v_h \in V_h.$$

Looking for the Solution...

If $u_h \in V_h$ is a solution to (V_h), then **in particular**

$$a(u_h', \phi_j') = (f, \phi_j), \quad j = 1, 2, \ldots, N.$$

Also, we can write u_h in terms of the basis functions:

$$u_h(x) = \sum_{j=1}^{N} \xi_j \phi_j(x), \quad \xi_j = u_h(x_j).$$

Therefore we can rewrite the above equation

$$\sum_{i=1}^{N} \xi_i a(\phi_i', \phi_j') = (f, \phi_j), \quad j = 1, 2, \ldots, N.$$

The Stiffness Matrix and Load Vector

From

$$\sum_{i=1}^{N} \xi_i a(\phi_i', \phi_j') = (f, \phi_j), \quad j = 1, 2, \ldots, N$$

we identify the vectors $\tilde{\xi} = \{\xi_1, \xi_2, \ldots, \xi_N\}^T$, and $\tilde{\mathbf{b}} = \{b_1, b_2, \ldots, b_N\}^T$, where $b_i = (f, \phi_i)$, and the matrix A, where $A_{ij} = a(\phi_i', \phi_j')$.

We can now write our problem as

$$A\tilde{\xi} = \tilde{\mathbf{b}}.$$

For historical reasons (Structural Mechanics) A is known as the **stiffness matrix**, and $\tilde{\mathbf{b}}$ as the **load vector**.

Computing the Stiffness Matrix

The elements of the stiffness matrix can be computed: —

First we notice that if the basis is the piecewise linear "tent-functions", then if $|i-j|>1$, then ϕ_i and ϕ_j are non-overlapping which means $a(\phi_i', \phi_j') = 0$.

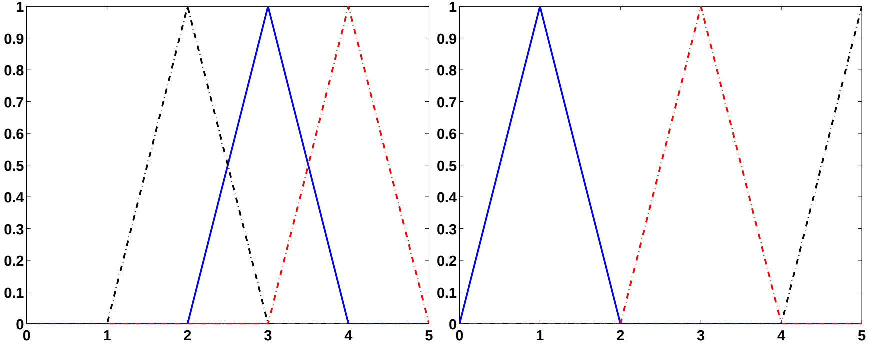
The Diagonal Entries, $j = 1, 2, \dots, N$.

$$a(\phi_j', \phi_j') = \int_{x_{j-1}}^{x_j} \frac{1}{h_j^2}\,dx + \int_{x_j}^{x_{j+1}} \frac{1}{h_{j+1}^2}\,dx = \frac{1}{h_j} + \frac{1}{h_{j+1}}$$

The Super- and Sub-Diagonal Entries, $j = 2, 3, \dots, N$.

$$a(\phi_{j-1}', \phi_j') = a(\phi_j', \phi_{j-1}') = \int_{x_{j-1}}^{x_j} \frac{-1}{h_j}\frac{1}{h_j}\,dx = -\frac{1}{h_j}$$

Illustration: Overlapping and Nonoverlapping Tent-Functions



The Stiffness Matrix is Symmetric Positive Definite (SPD)

A is **symmetric** since $a(u,v) = a(v,u)$, and with $v(x) = \sum_{i=1}^{N} \eta_j \phi_i(x)$ we get

$$\sum_{i,j=1}^{N} \eta_i a(\phi_i', \phi_j')\eta_j = a\left(\sum_{i=1}^{N} \eta_i \phi_i', \sum_{j=1}^{N} \eta_j \phi_j'\right) = \int_I [v_h'(x)]^2\, dx \geq 0$$

Equality holds if and only if $v'(x) \equiv 0$, which implies $v(x) \equiv 0$ by the boundary conditions.

Fact: A matrix A is SPD if

$$x^T A x > 0, \quad \forall x \in \mathbb{R}^n \setminus \{0\}$$

Fact: An SPD matrix *(i)* has positive eigenvalues, *(ii)* is non-singular.

The Special Case $h_j = h,\ j = 1, 2, \ldots, N$

If we equi-partition the interval we get the linear system

$$\frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & \ldots & \ldots & 0 \\ -1 & 2 & -1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & -1 & 2 & -1 \\ 0 & \ldots & \ldots & 0 & -1 & 2 \end{bmatrix} \tilde{\xi} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ \vdots \\ b_N \end{bmatrix}$$

If we divide by h the we recover the standard **finite difference method** for the problem, where the right hand-side

$$b_j = \int_{x_{j-1}}^{x_{j+1}} f(x)\phi_j \, dx$$

is a **weighted mean** of $f(x)$ over the interval $[x_{j-1}, x_{j+1}]$.

Estimating the Error for the Model Problem

We are now going to look at the error $(\mathbf{u} - \mathbf{u_h})$ where u is the exact solution of the differential equation, and u_h is the solution to the finite element problem (V_h).

Since $V_h \subset V$,

$$
\begin{array}{l}
a(u', v_h') = (f, v_h), \quad \forall v_h \in V_h \\
a(u_h', v_h') = (f, v_h), \quad \forall v_h \in V_h \\
\hline
a((u - u_h)', v_h') = 0, \quad \forall v_h \in V_h
\end{array}
$$

That means that the error is orthogonal to the function space V_h (as measured by $a(\circ', \circ')$.)

Definition and Tools

Definition (The L_2-norm ($L_{2,a}$?))

$$\|w\|_2 = \sqrt{a(w,w)} = \sqrt{\int_0^1 w^2\,dx}$$

Theorem (Cauchy's Inequality)

$$|a(v,w)| \leq \|v\|\|w\|$$

Cauchy's Inequality: Proof

Cauchy's Inequality.

Given v and w, define the renormalized functions $\widehat{v} = \frac{v}{\|v\|}$ and $\widehat{w} = \frac{w}{\|w\|}$. Which means $\|\widehat{v}\| = \|\widehat{w}\| = 1$. Now

$$\begin{aligned} 0 &\leq \| \pm \widehat{v} - \widehat{w}\|^2 = a(\pm\widehat{v} - \widehat{w}, \pm\widehat{v} - \widehat{w}) \\ &= a(\widehat{v}, \widehat{v}) \mp 2a(\widehat{v}, \widehat{w}) + a(\widehat{w}, \widehat{w}) \\ &= 2 \mp 2a(\widehat{v}, \widehat{w}). \end{aligned}$$

Hence,

$$|a(\widehat{v}, \widehat{w})| \leq 1.$$

Removing the linear normalization factors give

$$|a(v, w)| \leq \|v\|\|w\|.$$

$\square$

Error Control — Theorem

Theorem

For any $v_h \in V_h$ we have

$$\|(u - u_h)'\| \leq \|(u - v_h)'\|.$$

So, measured in the L_2-norm of the derivative (also known as the H^1-seminorm), the solution u_h to the discrete problem is closer to the solution u to the original continuous ODE than any other function in the function space V_h.

Error Control — Proof

Proof.

Let $v_h \in V_h$ be arbitrary, and set $w_h = u_h - v_h$. Then $w_h \in V_h$ and we get:

$$\begin{aligned}
\|(u-u_h)'\|^2 &= a((u-u_h)',(u-u_h)') + \underbrace{a((u-u_h)',w_h')}_{=0} \\
&= a((u-u_h)',(u-u_h+w_h)') \\
&= a((u-u_h)',(u-v_h)') \\
\{\text{Cauchy's}\} &\leq \|(u-u_h)'\|\|(u-v_h)'\|.
\end{aligned}$$

Dividing through by $\|(u-u_h)'\|$ gives

$$\|(u-u_h)'\| \leq \|(u-v_h)'\|.$$

□

Error Control — Applying the Theorem

From the theorem we can get a quantitative estimate (upper bound) for the error $\|(u-u_h)'\|$ by estimating $\|(u-v_h)'\|$ where $v_h \in V_h$ is a suitably chosen function.

Let v_h be the **linear interpolant** of u, *i.e.* $v_h(x_j) = u(x_j)$, then

$$|u'(x) - v_h'(x)| \leq h \max_{x\in[0,1]} |u''(x)| \quad [1]$$

$$|u(x) - v_h(x)| \leq \frac{h^2}{8} \max_{x\in[0,1]} |u''(x)| \quad [2]$$

Error Control — Applying the Theorem

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The first result and the theorem gives

$$\|(u-u_h)'\| \leq \mathcal{C}_1 h \max_{x\in[0,1]} |u''(x)|.$$

Similarly, from [2], we can get

$$\|u(x) - u_h(x)\| \leq \mathcal{C}_2 h^2 \max_{x\in[0,1]} |u''(x)|.$$

FEM for PDEs — the Poisson Equation

Consider the following boundary value problem

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \Gamma = \partial\Omega, \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^2 = \{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 \in \mathbb{R}\}$. Γ is the boundary of Ω, $f(x_1, x_2)$ a given function, and

$$\Delta u = \left[\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right] u(x_1, x_2) = u_{x_1 x_1}(x_1, x_2) + u_{x_2 x_2}(x_1, x_2).$$

Physical problems: Our simple 1-D model problems from last lecture carry over to this 2-D model: heat distribution in a plate, the displacement of an elastic membrane fixed at the boundary under transverse load, etc...

Vector Calculus — Review (or Crash Course) I/III

We need to generalize "integration by parts" to higher dimensions... We start with the **divergence theorem**:

$$\int_{\Omega} \nabla \circ \tilde{\mathbf{u}}\, d\tilde{\mathbf{x}} = \int_{\Gamma} \tilde{\mathbf{u}} \circ \tilde{\mathbf{n}}\, ds,$$

where $\tilde{\mathbf{u}} = \{u_1, u_2\}^T$, and

$$\nabla \circ \tilde{\mathbf{u}} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}.$$

$\tilde{\mathbf{n}} = \{n_1, n_2\}^T$ is the unit (length-one, outward) normal to Γ. $d\tilde{\mathbf{x}}$ denotes area-integration, and ds integration along the boundary. *c.f.* in 1-D:

$$\int_{[a,b]} \frac{du}{dx}\, dx = \int_{\{a,b\}} u(\mp 1)\, ds = -u(a) + u(b)$$

Vector Calculus — Review (or Crash Course) II/III

If we apply the divergence theorem to the vector functions $\tilde{\mathbf{u}}_1 = (vw, 0)$ and $\tilde{\mathbf{u}}_2 = (0, vw)$, we find

$$\int_\Omega \frac{\partial v}{\partial x_i} w + v \frac{\partial w}{\partial x_i} \, d\tilde{\mathbf{x}} = \int_\Gamma vwn_i \, ds, \qquad i = 1, 2.$$

Let ∇v denote the gradient

$$\nabla v = \left\{ \frac{\partial v}{\partial x_1}, \frac{\partial v}{\partial x_2} \right\}^T,$$

and use the result above to write **Green's Formula**

$$\begin{aligned}
\int_\Omega \nabla v \circ \nabla w \, d\tilde{\mathbf{x}} &\equiv \int_\Omega \left[\frac{\partial v}{\partial x_1} \frac{\partial w}{\partial x_1} + \frac{\partial v}{\partial x_2} \frac{\partial w}{\partial x_2} \right] d\tilde{\mathbf{x}} \\
&= \int_\Gamma \left[v \frac{\partial w}{\partial x_1} n_1 + v \frac{\partial w}{\partial x_2} n_2 \right] ds - \int_\Omega v \left[\frac{\partial^2 w}{\partial x_1^2} + \frac{\partial^2 w}{\partial x_2^2} \right] d\tilde{\mathbf{x}} \\
&= \int_\Gamma v \frac{\partial w}{\partial n} \, ds - \int_\Omega v \Delta w \, d\tilde{\mathbf{x}}.
\end{aligned}$$

Vector Calculus — Review (or Crash Course) III/III

The normal derivative

$$\frac{\partial w}{\partial n} = \frac{\partial w}{\partial x_1} n_1 + \frac{\partial w}{\partial x_2} n_2 = \nabla w \circ \tilde{\mathbf{n}}$$

is the derivative in the outward normal direction to the boundary Γ.

With this notation we are ready to state the variational formulation corresponding to the differential equation...

Variational Formulation

$$(V) \qquad \text{Find } u \in V \text{ so that } a(u,v) = (f,v),\ \forall v \in V.$$

where

$$a(u,v) = \int_\Omega \nabla u \circ \nabla v \, d\tilde{\mathbf{x}}$$

$$(f,v) = \int_\Omega fv \, d\tilde{\mathbf{x}}$$

$$V = \left\{ \begin{array}{ll} v: & v \in C(\Omega) \\ & v_{x_1} \text{ and } v_{x_2} \text{ are piecewise continuous in } \Omega \\ & v = 0 \text{ on } \Gamma \end{array} \right\}$$

Note: We have slightly changed the notation of the $a(\circ,\circ)$ inner-product. This is very common in the PDE framework. The $a(\circ,\circ)$ inner-product generically involves integrating the product(s) of derivative(s) of the two arguments over the domain.

Minimization Formulation

u is a solution to (V) if and only if it is also a solution to the following minimization problem.

$$(M) \qquad \text{Find } u \in V \text{ such that } F(u) \leq F(v) \; \forall v \in V,$$

where $F(v)$ is the potential energy

$$F(v) = \frac{1}{2} a(v, v) - (f, v).$$

Next...

Constructing a Finite Element Method for this problem...