Numerical Solutions to Differential Equations

Lecture Notes FEM#3

The Finite Element Method — Triangulations, the Element Stiffness Matrix; Hilbert Spaces; Geometric Interpretation

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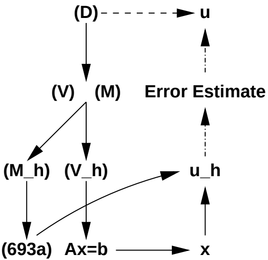
Outline

Quick Recap, Our View of the FEM Problem



We are seeking the solution u to the differential equation (D). We formulate the equivalent variational (V) and/or minimization (M) problems. Next, we construct a finite dimensional variational problem (V_h), which leads us to a linear system $Ax = b$.

The solution $x = A^{-1}b$ is a solution to (V_h). The error estimate gives us an idea of how close out numerical solution is to the solution of (V).

Quick Recap, the Poisson Equation

We are trying to solve the following boundary value problem

$$\begin{aligned} -\Delta u &= f && \text{in } \Omega \\ u &= 0 && \text{on } \Gamma = \partial\Omega \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^2 = \{(x_1, x_2) : x_1 \in \mathbb{R}, x_2 \in \mathbb{R}\}$.
Γ is the boundary of Ω, $f(x_1, x_2)$ a given function.
We reviewed some vector calculus, *e.g.* the **divergence theorem**

$$\int_\Omega \nabla \circ \tilde{\mathbf{u}}\, d\tilde{\mathbf{x}} = \int_\Gamma \tilde{\mathbf{u}} \circ \tilde{\mathbf{n}}\, ds,$$

and **Green's Formula**

$$\int_\Omega \nabla v \circ \nabla w\, d\tilde{\mathbf{x}} = \int_\Gamma v \frac{\partial w}{\partial n}\, ds - \int_\Omega v \Delta w\, d\tilde{\mathbf{x}}.$$

Quick Recap, the Variational Formulation

We defined the equivalent variational problem

$$(V) \qquad \text{Find } u \in V \text{ so that } a(u,v) = (f,v),\ \forall v \in V.$$

where

$$a(u,v) = \int_\Omega \nabla u \circ \nabla v \, d\tilde{\mathbf{x}}$$

$$(f,v) = \int_\Omega fv \, d\tilde{\mathbf{x}}$$

$$V = \left\{ \begin{array}{ll} v: & v \in C(\Omega) \\ & v_{x_1} \text{ and } v_{x_2} \text{ are piecewise continuous in } \Omega \\ & v = 0 \text{ on } \Gamma \end{array} \right\}$$

Now, we will construct a finite-element subspace $V_h \subset V$ and set up the Finite Element Method for this problem.

Step #1: Triangulating the Domain Ω

For simplicity, we assume that the boundary Γ of Ω is a polygonal curve. [Otherwise we have to worry about how well the polygons approximate the boundary and/or use non-polygonal boundary elements...]

We generate a **triangulation** by subdividing Ω into a set $T_h = \{t_1, t_2, \dots, t_N\}$ of non-overlapping triangles. $\Omega = \bigcup_{i=1}^{N} t_i$. Note that no vertex of one triangle lies on the edge of another triangle.

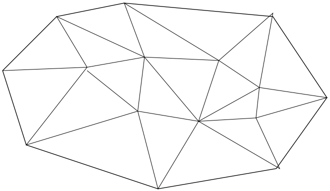


Figure: A sample finite element triangulation of a polygonal domain.

Size of the Triangulation / The Finite Element Space

We need a measure of the grid size. Let

$$\mathrm{diam}_{2\mathrm{D}}(t) = \text{the longest side of the triangle } t.$$

and

$$h = \max_{t \in T_h} \mathrm{diam}_{2\mathrm{D}}(t)$$

We are now ready to define the finite element space V_h

$$V_h = \left\{ v : \begin{array}{l} v \in C(\Omega) \\ v\big|_t \text{ is linear for } t \in T_h \\ v = 0 \text{ on } \Gamma \end{array} \right\}$$

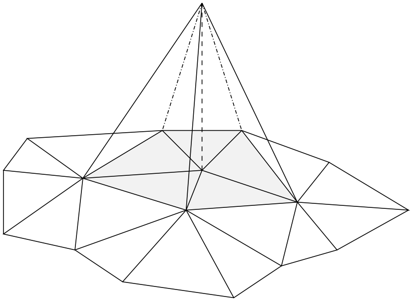
where $v\big|_t$ denotes the restriction of v to the triangle t.

Clearly $V_h \subset V$.

Basis Functions for V_h

The parameters which describe the functions of V_h are the values at the node points n_j of $v \in V_h$ (excluding nodes on the boundary, since $v = 0$ there). The corresponding basis functions $\phi_i \in V_h$, $i = 1, 2, \ldots, N$ are defined by

$$\phi_i(n_j) = \delta_{ji}, \quad i, j = 1, 2, \ldots, N$$



The Basis Functions ϕ_j

We see that the support of ϕ_j — the set of points $\tilde{\mathbf{x}}$ for which $\phi_j(\tilde{\mathbf{x}}) \neq 0$ — are the union of the triangles with the common node n_j.

We can now represent any function $v_h \in V_h$:

$$v_h(\tilde{\mathbf{x}}) = \sum_{i=1}^{N} \eta_j \phi_j(\tilde{\mathbf{x}}), \quad \eta_j = \phi_j(n_j), \quad \forall \tilde{\mathbf{x}} \in \Omega$$

We are now ready for the finite element formulation of the problem!

Finite Element Formulation

The discrete Variational Problem is

$$(V_h) \qquad \text{Find } u_h \in V_h \text{ such that } a(u_h, v_h) = (f, v_h),\ \forall v_h \in V_h.$$

As in the 1-dimensional case, we plug in the basis-representation of u_h:

$$u_h(\tilde{\mathbf{x}}) = \sum_{i=1}^{N} \xi_i \phi_i(\tilde{\mathbf{x}})$$

and notice that (V_h) implies that $a(u_h, \phi_j) = (f, \phi_j)$, $j = 1, 2, \ldots, N$. Thus

$$\sum_{i=1}^{N} \xi_i a(\phi_i, \phi_j) = (f, \phi_j)$$

defines the linear system

$$A\tilde{\xi} = \tilde{\mathbf{b}}$$

The Linear System $A\tilde{\xi} = \tilde{\mathbf{b}}$

The entries of the stiffness matrix A are

$$A_{i,j} = a(\phi_i, \phi_j) = \int_\Omega \nabla\phi_i \circ \nabla\phi_j \, d\tilde{\mathbf{x}} = \sum_{t \in T_h} \left[\int_t \nabla\phi_i \circ \nabla\phi_j \, d\tilde{\mathbf{x}} \right]$$

The entries of the load vector $\tilde{\mathbf{b}}$ are

$$b_j = (f, \phi_j) = \int_\Omega f(\tilde{\mathbf{x}})\phi_j(\tilde{\mathbf{x}}) \, d\tilde{\mathbf{x}} = \sum_{t \in T_h} \left[\int_t f(\tilde{\mathbf{x}})\phi_j(\tilde{\mathbf{x}}) \, d\tilde{\mathbf{x}} \right]$$

Computing these entries is not completely trivial...

Notes on the Linear System

We note that A is symmetric, and positive definite since

$$a(v,v) = 0 \Rightarrow \nabla v = 0 \Rightarrow v \equiv 0, \text{ and } a(v,v) > 0, \ \forall v \not\equiv 0.$$

Since A is SPD, $\tilde{\xi} = A^{-1}\tilde{\mathbf{b}}$ has a unique solution.

Also, if n_i and n_j are **not** nodes of the same triangle, we have

$$a(\phi_i, \phi_j) = 0$$

The Element Stiffness Matrix

In practice, the elements A_{ij} are usually computed by summing the contributions from the different triangles:

$$A_{i,j} = a(\phi_i, \phi_j) = \sum_{t \in T_h} a|_t(\phi_i, \phi_j)$$

where $a|_t(\phi_i, \phi_j)$ is $a(\phi_i, \phi_j)$ restricted to the triangle t, *i.e.*

$$a|_t(\phi_i, \phi_j) = \int_t \nabla \phi_i(\tilde{\mathbf{x}}) \circ \nabla \phi_j(\tilde{\mathbf{x}})\, d\tilde{\mathbf{x}}$$

Let n_i, n_j and n_k denote the vertexes of the triangle t, then we call the 3×3 matrix $a|_t$ is called the **element stiffness matrix for** t

$$a|_t = \begin{bmatrix} a|_t(\phi_i, \phi_i) & a|_t(\phi_i, \phi_j) & a|_t(\phi_i, \phi_k) \\ a|_t(\phi_j, \phi_i) & a|_t(\phi_j, \phi_j) & a|_t(\phi_j, \phi_k) \\ a|_t(\phi_k, \phi_i) & a|_t(\phi_k, \phi_j) & a|_t(\phi_k, \phi_k) \end{bmatrix}$$

Assembling the Global Stiffness Matrix

First we compute $a|_t$, $\forall\, t \in T_h$; then we can assemble the global stiffness matrix by summing the contributions from each sub-triangle.

In a similar way, we can assemble the right-hand side $\tilde{\mathbf{b}}$.

In computing $a|_t$ we are need the basis functions ϕ_i, ϕ_j and ϕ_k, restricted to the triangle t. Let ψ_i, ψ_j and ψ_k denote these restrictions. Clearly $\psi_{i,j,k}$ satisfy the following at the nodes:

$$\begin{array}{lll} \psi_i(n_i) = 1, & \psi_i(n_j) = 0, & \psi_i(n_k) = 0 \\ \psi_j(n_i) = 0, & \psi_j(n_j) = 1, & \psi_j(n_k) = 0 \\ \psi_k(n_i) = 0, & \psi_k(n_j) = 0, & \psi_k(n_k) = 1 \end{array}$$

ψ_i, ψ_j and ψ_k are the **basis functions on the triangle** t^{ijk}.

The Basis Functions on the Triangle t

Let us compute the basis functions on the triangle t^{ijk}, with nodes

$$n_i = (x_i, y_i), \quad n_j = (x_j, y_j), \quad n_k = (x_k, y_k).$$

They are linear functions

$$\psi_i = a_i + b_i x + c_i y, \; \psi_j = a_j + b_j x + c_j y, \; \psi_k = a_k + b_k x + c_k y,$$

and we have the following relations

$$\begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix} \begin{bmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence,

$$\begin{bmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix} = \begin{bmatrix} 1 & x_i & y_i \\ 1 & x_j & y_j \\ 1 & x_k & y_k \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example: Basis Functions for $\{ (0,0), (0,1), (1,0) \}$ I/II

For the triangle described by the nodes

$$n_i = (0,0), \quad n_j = (0,1), \quad n_k = (1,0)$$

We get

$$\begin{bmatrix} a_i & a_j & a_k \\ b_i & b_j & b_k \\ c_i & c_j & c_k \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

i.e. the basis functions are

$$\psi_i(x,y) = 1 - x - y, \quad \psi_j(x,y) = y, \quad \psi_k(x,y) = x$$

their gradients are

$$\nabla\psi_i(x,y) = \{-1,-1\}^T, \; \nabla\psi_j(x,y) = \{0,1\}^T, \; \nabla\psi_k(x,y) = \{1,0\}^T$$

Example: Basis Functions for $\{ (0,0), (0,1), (1,0) \}$

The gradients are

$$\nabla\psi_i(x,y) = \{-1,-1\}^T,\ \nabla\psi_j(x,y) = \{0,1\}^T,\ \nabla\psi_k(x,y) = \{1,0\}^T$$

The inner product of the gradients:

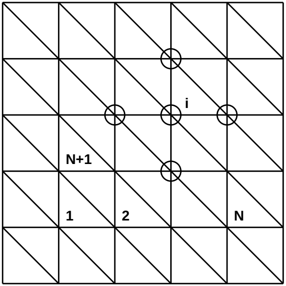
	$\nabla\psi_i$	$\nabla\psi_j$	$\nabla\psi_k$
$\nabla\psi_i$	2	−1	−1
$\nabla\psi_j$	−1	1	0
$\nabla\psi_k$	−1	0	1

The element stiffness matrix:

$$a\big|_t = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}$$

Example: A Square Domain

Let Ω be the unit square, and let T_h be the uniform triangulation shown in the figure below, with the indicated numbering of the nodes.



For this triangulation we get (row $\#i$ of the stiffness matrix, n_i does not have a neighbor on the boundary)

$$A_{i,i} = 4, \quad A_{i,i+1} = A_{i,i-1} = -1, \quad A_{i,i+N} = A_{i,i-N} = -1$$

Moving to Hilbert Spaces

When solving BVPs for PDEs, it is natural and useful (from a mathematical perspective) to work with function spaces that contain more functions than the continuous functions with piecewise continuous derivatives.

Our new spaces will also have associated **scalar products** related to the BVP.

Our expanded spaces will be **Hilbert Spaces**.

What's a Hilbert Space? — Let's Get Technical!

"A Hilbert Space is a complete metric space, i.e. every Cauchy sequence converges with respect to the norm induced by the metric."

Let's start from the beginning...

Definition (Linear form)

If V is a linear space, then L is a linear form on V if

$$L : V \to \mathbb{R}, \text{ i.e. } L(v) \in \mathbb{R},\ \forall v \in V, \quad \text{and } L \text{ is linear}$$

Linearity means that $\forall v, w \in V$ and $\alpha, \beta \in \mathbb{R}$

$$L(\alpha v + \beta w) = \alpha L(v) + \beta L(w)$$

Some More Definitions

Definition (Bilinear Form)

A form $a(\circ, \circ)$ is a bilinear form on $V \times V$ if

$$a : V \times V \to \mathbb{R}, \quad \text{i.e. } a(v, w) \in \mathbb{R}, \ \forall v, w \in V$$

and $a(\circ, \circ)$ is linear in each argument

$$\left.\begin{array}{rcl} a(u, \alpha v + \beta w) & = & \alpha a(u, v) + \beta a(u, w) \\ a(\gamma u + \delta v, w) & = & \gamma a(u, w) + \delta a(v, w) \end{array}\right\} \quad \begin{array}{l} \forall u, v, w \in V \\ \forall \alpha, \beta, \gamma, \delta \in \mathbb{R} \end{array}$$

The bilinear form $a(\circ, \circ)$ on $V \times V$ is said to be **symmetric** if

$$a(u, v) = a(v, u), \quad \forall u, v \in V$$

Bilinear From → Scalar Product → Norm

A symmetric bilinear form $a(\circ,\circ)$ on $V \times V$ is said to be a **scalar product** on V if

$$a(v,v) > 0, \quad \forall v \in V \backslash \{0\}$$

The **norm** $\| \circ \|_a$ associated with the scalar product $a(\circ,\circ)$ is defined by

$$\|v\|_a = \sqrt{a(v,v)}, \quad \forall v \in V$$

Cauchy's Inequality

$$|a(v,w)| \leq \|v\|_a \|w\|_a$$

holds.

Cauchy Sequence → Hilbert Space

A sequence in the space V with norm $\| \circ \|$

$$\{v_k\}_{k=1}^{\infty}, \quad v_k \in V, \; k = 1, 2, \ldots, \infty$$

is said to be a **Cauchy sequence** if $\forall\, \epsilon > 0$ there is an $N(\epsilon)$ so that

$$\|v_i - v_j\| < \epsilon, \quad \forall i, j > N(\epsilon)$$

Further, the sequence converges to $v \in V$ if

$$\lim_{i \to \infty} \|v_i - v\| = 0$$

If V is a linear space with a norm $\| \circ \|$ and *every* Cauchy sequence with respect to $\| \circ \|$ is convergent, then V is a **Hilbert Space**.

Hilbert Spaces — Examples: $\mathbf{L_2(I)}$

If $I = (a, b)$ is an interval, we define the space of "square integrable functions" on I:

$$L_2(I) = \left\{ v : \; v \text{ is defined on } I \text{ and } \int_I v^2 \, dx < \infty \right\}$$

$L_2(I)$ is a Hilbert space with the scalar product and norm

$$(u, v) = \int_I uv \, dx, \qquad \|v\|_{L_2(I)} = \sqrt{\int_I v^2 \, dx} = \sqrt{(v, v)}$$

By Cauchy's inequality

$$|(u, v)| \leq \|u\|_{L_2(I)} \|v\|_{L_2(I)}$$

We see that (u, v) exists as long as $u, v \in L_2(I)$.

The Hilbert Space $\mathbf{L_2(I)}$

$L_2(I)$ is a very "rich" space — for full appreciation we need familiarity with the Lebesgue integral (a basic course in Real Analysis). Here it suffices to think of $L_2(I)$-functions as piecewise continuous, possibly unbounded, with finite square-integral.

Example: For $I = (0, 1)$, the functions $v(x) = x^{-\beta}$, $\beta < 1/2$ are members if $L_2(0, 1)$ since:

$$\int_0^1 \left[x^{-\beta}\right]^2 dx = \int_0^1 \left[x^{-2\beta}\right] dx = \left[\frac{x^{1-2\beta}}{1-2\beta}\right]_0^1 = \frac{1}{1-2\beta} < \infty$$

as long as $\beta < 1/2$.

We have already worked with $L_2(\Omega)$ — even though we didn't "know" we did... All our Finite Element Solutions where in the space(s) $L_2(\Omega)$.

The Hilbert Space $\mathbf{L_2(I)}$ — A Side Note

One very important property of functions in $L_2(\mathbb{R})$ is that the **Fourier transform**

$$\widehat{f}(w) = \mathcal{F}[f](w) = \int_{-\infty}^{\infty} f(t)e^{-iwt}\,dx$$

$$f(t) = \mathcal{F}^{-1}[\widehat{f}](t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(w)e^{iwt}\,dt$$

is an isometry on $L_2(\mathbb{R})$. — *i.e.* if $f(t) \in L_2(\mathbb{R})$, then so is its Fourier transform, $\widehat{f}(w) \in L_2(\mathbb{R})$.

We need "Smoother" Function Spaces

As we have seen, functions in $L_2(I)$ can be quite "wild" — $x^{-1/3}$ approaches ∞ as $x \to 0$, but $x^{-1/3} \in L_2(0,1)$.

The point of looking at Hilbert spaces is that we are going to look for solutions to our ODEs/PDEs (the Finite Element formulation) in these spaces... We need something a bit "smoother"...

$H^1(I)$ is the space of square-integrable functions, with square-integrable derivatives...

The Sobolev Space $H^1(I)$

If $I = (a, b)$ is an interval, we define

$$H^1(I) = \left\{v : \ v \in L_2(I), \ v' \in L_2(I)\right\}$$

$H^1(I)$ is known as a Sobolev space — if we define the scalar product and norm

$$(u, v)_{H^1(I)} = \int_I uv + u'v' \, dx$$

$$\|v\|_{H^1(I)} = \sqrt{\int_I v^2 + v'^2 \, dx} = \sqrt{(v, v)_{H^1(I)}}$$

it is a Hilbert space.

In order for a function of the form $x^{-\beta}$ to be a member of $H^1(0, 1)$ we must have $\beta < 1/2$ and $(\beta + 1) < 1/2 \Rightarrow \beta < -1/2$. This means that even $\sqrt{x}$ is too "wild" for $H^1(0, 1)$.

A Space Useful for Boundary Value Problems: $\mathbf{H_0^1(I)}$

Whereas we may admire Hilbert spaces for their mathematical beauty... We are on a mission of solving boundary value problems.

For a problem of the form:

$$-u''(x) = f(x),\ x \in I = (a,b), \quad u(a) = u(b) = 0$$

The space

$$H_0^1(I) = \{v \in H^1(I) :\ v(a) = v(b) = 0\}$$

with the scalar product and norm inherited from $H^1(I)$ is very useful.

Reformulating the Variational Problem

Our model boundary value problem

$$\text{(BVP)} \qquad -u''(x) = f(x), \;\; x \in [0,1], \quad u(0) = u(1) = 0$$

can now be given the following variational formulation

$$(V) \qquad \text{Find } u \in H_0^1(I) \text{ such that } a(u,v) = (f,v),\; \forall v \in H_0^1(I).$$

where

$$a(u,v) = (u', v') = \int_I u' v' \, dx, \quad (f,v) = \int_I fv \, dx$$

Comments

We notice that $H_0^1(I)$ is larger than the space of piecewise linear functions on $[0, 1]$ (which we used in our previous variation formulations).

$H_0^1(I)$ is optimally tailored for the variational formulation (V) of (BVP); — it is the largest space for which (V) is meaningful.

Since $u, v \in H_0^1(I)$, Cauchy's inequality guarantees that $a(u, v)$ is meaningful (finite):

$$|a(u, v)| = |(u', v')| \leq \|u'\|_{L_2(I)} \|v'\|_{L_2(I)} < \infty$$

and $|(f, v)|$ is also finite as long as $f \in L_2(I)$:

$$|(f, v)| \leq \|f\|_{L_2(I)} \|v\|_{L_2(I)}$$

More Comments

From a mathematical point of view, the "right" (optimal) choice of function space makes it easier to prove existence and uniqueness of the solutions of the continuous variational problem.

From a finite element point of view the "$H_0^1(I)$" formulation is mainly of interest since the basic error estimate

$$\|(u - u_h)'\| \leq \|(u - v)'\|$$

can expressed in terms of the $H_1(I)$-norm.

$$\|(u - u_h)'\| \leq \|(u - v)'\| \leq \|u - v\|_{H_0^1(I)}$$

Using the standard (mathematical) notation $L_2(I)$, $H^1(I)$, $H_0^1(I)$ etc enables us to express the variational formulations in a concise way.

Further Comments, Moving Beyond 1D-problems

Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geq 2$, and define

$$L_2(\Omega) = \left\{ v : \; v \text{ is defined on } \Omega \text{ and } \int_\Omega v^2 \, d\tilde{\mathbf{x}} < \infty \right\}$$

$$H^1(\Omega) \left\{ v : \; v \in L_2(\Omega), \; \frac{\partial v}{\partial x_i} \in L_2(\Omega), \; i = 1, 2, \dots, n \right\}$$

and introduce the corresponding scalar products and norms

$$(u, v) = \int_\Omega uv \, d\tilde{\mathbf{x}}, \quad \|u\|_{L_2(\Omega)} = \sqrt{\int_\Omega v^2 \, d\tilde{\mathbf{x}}}$$

$$(u, v)_{H^1(\Omega)} = \int_\Omega [uv + \nabla u \circ \nabla v] \, d\tilde{\mathbf{x}}, \quad \|u\|_{H^1(\Omega)} = \sqrt{(u, u)_{H^1(\Omega)}}$$

we also define

$$H_0^1(\Omega) = \left\{ v : \; v \in H^1(\Omega), \; v = 0 \text{ on } \Gamma \right\}, \quad \text{where } \Gamma = \partial\Omega.$$

Completing the Circle

The Boundary Value Problem

$$\text{(BVP)} \qquad -\Delta u = f,\ x \in \Omega, \quad u = 0,\ x \in \Gamma = \partial\Omega$$

can now be given the variational formulation

$$\text{(V)} \qquad \text{Find } u \in H_0^1(\Omega) \text{ such that } a(u,v) = (f,v),\ \forall v \in H_0^1(\Omega)$$

or equivalently

$$\text{(M)} \qquad \text{Find } u \in H_0^1(\Omega) \text{ such that } F(u) \leq F(v),\ \forall v \in H_0^1(\Omega)$$

where

$$F(v) = \frac{1}{2}a(v,v) - (f,v)$$

$$a(u,v) = \int_\Omega \nabla u \circ \nabla v \, d\tilde{\mathbf{x}}, \quad (f,v) = \int_\Omega fv \, d\tilde{\mathbf{x}}$$

"That's Weak!" In a Mathematical Sense, $\frac{I}{II}$

The formulation (V) is said to be a **weak formulation** of (BVP) and the solution of (V) is a **weak solution** of (BVP).

If u is a weak solution of (BVP) then **it is not** clear that u is also a classical solution of (BVP). For that to be true, we must require u to be sufficiently smooth that Δu us defined in a classical sense.

The mathematical advantage of the weak formulation is that it is easy to prove existence of a solution to (V), whereas it is relatively difficult to prove the existence of a classical solution to (BVP).

"That's Weak!" In a Mathematical Sense, $\frac{II}{II}$

To prove existence of a classical solution of (BVP) we start with a weak solution and show — with considerable effort — that this solution is smooth enough to be a classical solution.

For more complicated (*e.g.* nonlinear) problems it may be extremely difficult or practically impossible to prove existence of classical solutions whereas existence of weak solutions may still be within reach.

Example: Inhomogeneous Boundary Conditions

I/II

Consider

$$\text{(BVP)} \qquad -\Delta u = f,\; x \in \Omega, \quad u = g(x), x \in \Gamma$$

Let

$$\mathbf{H^1_{g(x)}(\Omega)} = \{v :\; v \in H^1(\Omega),\; v(x) = g(x),\; x \in \Gamma\}$$

We multiply (BVP) by $v \in \mathbf{H^1_0(\Omega)}$ and integrate

$$-\int_\Omega v \Delta u \, d\tilde{\mathbf{x}} = \int_\Omega vf \, d\tilde{\mathbf{x}}$$

We apply Green's Theorem to the left-hand side

$$-\int_\Omega v \Delta u \, d\tilde{\mathbf{x}} = \int_\Omega \nabla v \circ \nabla u \, d\tilde{\mathbf{x}} - \int_\Gamma v \frac{\partial u}{\partial n} \, d\tilde{\mathbf{s}}$$

Note that the boundary integral is **zero** since $v \in H^1_0(\Omega)$.

Example: Inhomogeneous Boundary Conditions

II
II

Now, with

$$a(u,v) = \int_\Omega \nabla u \circ \nabla v \, d\tilde{\mathbf{x}}, \quad (f,v) = \int_\Omega fv \, d\tilde{\mathbf{x}}$$

the corresponding variational and minimization formulations are

(V) Find $u \in H^1_{g(x)}(\Omega)$ such that $a(u,v) = (f,v)$, $\forall v \in H^1_0(\Omega)$

and

(M) Find $u \in H^1_{g(x)}(\Omega)$ such that $F(u) \leq F(v)$, $\forall v \in H^1_0(\Omega)$

where, as usual

$$F(v) = \frac{1}{2} a(v,v) - (f,v)$$

Note: A finite element representation of this problem yields finite elements with non-zero basis functions at the boundary...

Geometric Interpretation of FEM

Recall: Two elements u, v in a linear space S with scalar product $(\circ, \circ)$ are **orthogonal** if $(u, v) = 0$.

As a point of discussion, let us consider the following equation (from Mechanics)

$$\text{(BVP)} \quad -\Delta u + u = f, \; x \in \Omega, \quad u = 0, \; x \in \Gamma$$

The corresponding variational formulation is

$$\text{(V)} \quad \text{Find } u \in H_0^1(\Omega)\text{:} \; \underbrace{\int_\Omega [\nabla u \circ \nabla v + uv] \, d\tilde{\mathbf{x}}}_{a(u,v)} = \int_\Omega fv \, d\tilde{\mathbf{x}}, \; \forall v \in H_0^1(\Omega)$$

We notice that $a(\circ, \circ)$ is the $H_0^1(\Omega)$ scalar product.

Geometric Interpretation of FEM

In preparation for solving the FEM problem, we let V_h be a finite dimensional subspace of $H_0^1(\Omega)$, *e.g.* the space of piecewise linear functions on a triangulation of Ω.
The FEM formulation:

$$\text{(FEM)} \quad \text{Find } u_h \in V_h \text{ such that } a(u_h, v_h) = (f, v_h),\ \forall v_h \in V_h.$$

By subtracting

$$\begin{array}{rcll} a(u, v_h) & = & (f, v_h) & \forall v \in V_h \\ a(u_h, v_h) & = & (f, v_h) & \forall v \in V_h \\ \hline \mathbf{a([u - u_h], v_h)} & = & \mathbf{0} & \forall v \in V_h \end{array}$$

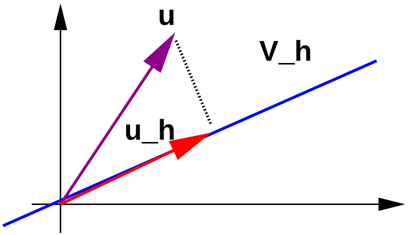
i.e. the error $[u - u_h]$ is orthogonal to the space V_h with respect to $a(\circ, \circ)$.

Geometric Interpretation of FEM

We can express the orthogonality $[u - u_h] \perp V_h$:

The finite element solution u_h is the **projection** *with respect to $a(\circ, \circ)$ of the exact solution u* **on** *V_h, i.e. u_h is the element in V_h closest to u with respect to the $H^1(\Omega)$-norm $\| \circ \|_{H^1(\Omega)}$, or*

$$\|u - u_h\|_{H^1(\Omega)} \leq \|u - v\|_{H^1(\Omega)}, \; \forall v \in V_h$$



Geometric Interpretation of FEM

According to

$$\|u - u_h\|_{H^1(\Omega)} \leq \|u - v\|_{H^1(\Omega)}, \ \forall v \in V_h$$

u_h is the best approximation of the exact solution u, in the sense that no other function $v \in V_h$, has a smaller error $[u - v]$ measured in the $H^1_0(\Omega)$-norm.

We have seen that u_h can be computed by solving a linear system of equations; — derived from the basis function representation of V_h.

Thus we can compute u_h — the best approximation of u — without knowing u itself.

Neumann Boundaries — Natural vs. Essential BCs

So far we have restricted our discussion to Dirichlet boundary conditions; we will now expand our universe to **Neumann problems**:

$$\text{(BVP)} \quad -\Delta u + u = f,\ x \in \Omega, \quad \frac{\partial u}{\partial \tilde{\mathbf{n}}} = g, x \in \Gamma$$

where as usual Ω is a bounded domain with boundary Γ, and $\frac{\partial}{\partial \tilde{\mathbf{n}}}$ denotes the outward normal derivative to Γ.

The Neumann boundary condition corresponds to a given force (mechanics) or a flow (physics) on Γ.

Variational Formulation for Neumann-BVP

In our usual fashion, we multiply (BVP) by a test function $v \in H^1(\Omega)$ and apply Green's theorem:

$$-\int_\Omega v \Delta u \, d\tilde{\mathbf{x}} = \int_\Omega \nabla u \circ \nabla \, d\tilde{\mathbf{x}} - \int_\Gamma v \frac{\partial u}{\partial \tilde{\mathbf{n}}} \, d\tilde{\mathbf{s}}$$

The variational formulation becomes

$$\text{(V)} \quad \text{Find } u \in H^1(\Omega)\text{:} \; a(u,v) = (f,v) + \langle g, v \rangle, \; \forall v \in H^1(\Omega)$$

where

$$a(u,v) = \int_\Omega \left[\nabla u \circ \nabla v + uv \right] \, d\tilde{\mathbf{x}}$$

$$(f,v) = \int_\Omega fv \, d\tilde{\mathbf{x}}, \quad \langle g, v \rangle = \int_\Gamma gv \, d\tilde{\mathbf{s}}$$

Variational Formulation for Neumann-BVP —- Comments

Note that the Neumann condition **does not** explicitly appear in the function space for the variational formulation; — we are only requiring $u \in H^1(\Omega)$. Still, any function solving (V) will satisfy the Neumann boundary condition.

A boundary condition of this type, which does not have to be explicitly imposed in the variational formulation, is called a **natural boundary condition**.

Boundary conditions (*e.g.* Dirichlet BCs) which have to be explicitly imposed are called **essential boundary conditions**.

Finite Element Method for the Neumann Problem

The FEM formulation for the Neumann problem, using piecewise linear basis functions:

Let T_h be a triangulation of Ω, as discussed earlier. Define

$$V_h = \{v : \ v \in C(\Omega), \ v|_t \text{ is linear } \forall t \in T_h\}$$

where the notation $v \in C(\Omega) \Leftrightarrow$ "*v is continuous on Ω*".

We characterize the functions in V_h by their values at the nodes, including nodes on the boundary Γ.

Note that function in V_h are not required to satisfy any boundary condition, and $V_h \subset H^1(\Omega)$.

Finite Element Method for the Neumann Problem

The FEM for (BVP)

(FEM) Find $u_h \in V_h$: $a(u_h, v_h) = (f, v_h) + \langle g, v_h \rangle, \; \forall v \in V_h$

where

$$a(u, v) = \int_\Omega [\nabla u \circ \nabla v + uv] \; d\tilde{\mathbf{x}}$$

$$(f, v) = \int_\Omega fv \; d\tilde{\mathbf{x}}, \quad \langle g, v \rangle = \int_\Gamma gv \; d\tilde{\mathbf{s}}$$

Using the "tent-function" basis we get a symmetric positive definite linear system $A\tilde{\xi} = \tilde{\mathbf{b}}$.

Finite Element Method for the Neumann Problem

As previously seen (for the Dirichlet problem) the solution u_h is optimal with respect to the $H^1(\Omega)$-norm, *i.e.*

$$\|u - u_h\|_{H^1(\Omega)} \leq \|u - v_h\|_{H^1(\Omega)}, \quad \forall v_h \in V_h$$

in particular we are allowed to choose v_h to be the interpolant of u, and get

$$\|u - u_h\|_{H^1(\Omega)} \leq \mathcal{C}h$$

Solved Problem: Mixing it up... $\frac{I}{IV}$

Let Ω be a bounded domain in $\mathbb{R}^2$ and let the boundary of Ω be divided into two parts Γ_1 and Γ_2.
(i) Give a variational formulation of the following problem:

$$\begin{aligned} -\Delta u &= f & x \in \Omega \\ u &= g & x \in \Gamma_1 \\ \frac{\partial u}{\partial \tilde{\mathbf{n}}} &= h & x \in \Gamma_2 \end{aligned}$$

where f, g, h are given functions.
(ii) Formulate a finite element method for this problem.

Solved Problem

Let

$$H^1_{\Gamma_1(g)} = \left\{ v \in H^1(\Omega) : \; v|_{\Gamma_1} = g \right\}$$

thus

$$H^1_{\Gamma_1(0)} = \left\{ v \in H^1(\Omega) : \; v|_{\Gamma_1} = 0 \right\}$$

Now multiply the equation by $v \in H^1_{\Gamma_1(0)}$ and apply Green's theorem to the left-hand-side:

$$-\int_\Omega v \Delta u \, d\tilde{\mathbf{x}} = \int_\Omega \nabla u \circ \nabla v \, d\tilde{\mathbf{x}} - \int_\Gamma v \frac{\partial u}{\partial \tilde{\mathbf{n}}} \, d\tilde{\mathbf{s}}$$

The integral over the Γ_1-part of the boundary is zero since v is zero there; over the Γ_2-part, we have $\frac{\partial u}{\partial \tilde{\mathbf{n}}} = h$ from the equation, hence

$$\int_\Gamma v \frac{\partial u}{\partial \tilde{\mathbf{n}}} \, d\tilde{\mathbf{s}} = \int_{\Gamma_2} vh \, d\tilde{\mathbf{s}}$$

Solved Problem

The variational formulation becomes

$$\text{(V)} \quad \text{Find } u \in H^1_{\Gamma_1(g)}(\Omega):\; a(u,v) = (f,v) + \langle h, v\rangle,\ \forall v \in H^1_{\Gamma_1(0)}(\Omega)$$

where

$$a(u,v) = \int_\Omega [\nabla u \circ \nabla v]\, d\tilde{\mathbf{x}}$$

$$(f,v) = \int_\Omega fv\, d\tilde{\mathbf{x}}, \quad \langle h, v\rangle = \int_{\Gamma_2} hv\, d\tilde{\mathbf{s}}$$

Let T_h be a triangulation of Ω, as discussed earlier. Define

$$V_h = \{v:\;\; v \in C(\Omega),\;\; v|_t \text{ is linear } \forall t \in T_h\}$$

$$W_h(g) = \{v:\;\; v \in V_h,\;\; v|_{\Gamma_1} = g\}$$

Solved Problem

Let T_h be a triangulation of Ω, as discussed earlier. Define

$$V_h = \{v : \ v \in C(\Omega), \ v|_t \text{ is linear } \forall t \in T_h\}$$

$$W_h(g) = \{v : \ v \in V_h, \ v|_{\Gamma_1} = g\}$$

The FEM for (BVP)

$$\text{Find } u_h \in W_h(g): \ a(u_h, v_h) = (f, v_h) + \langle h, v_h \rangle, \ \forall v \in W_h(0)$$

where

$$a(u, v) = \int_\Omega [\nabla u \circ \nabla v] \, d\tilde{\mathbf{x}}$$

$$(f, v) = \int_\Omega fv \, d\tilde{\mathbf{x}}, \quad \langle h, v \rangle = \int_{\Gamma_2} hv \, d\tilde{\mathbf{s}}. \qquad \square$$

Notes on Implementation of FEM

When developing a piece of software for the solution of FEM problems the process naturally divides into 6 steps:

[1] Input of Domain, Boundary Conditions, and the Equation.

[2] Triangulation, T_h, of the domain Ω.

[3] Computation of the element stiffness matrix, and element load vector.

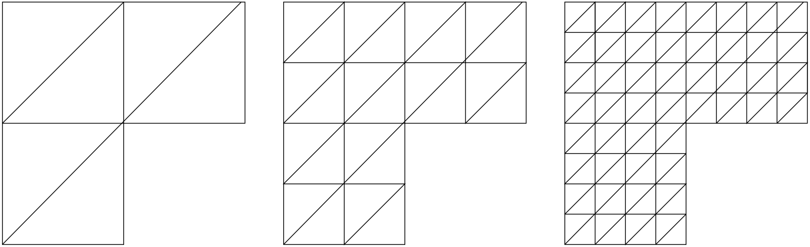
[4] Assembly of the global stiffness matrix, and the load vector.

[5] Solution of the linear system.

[6] Presentation of the result.

Some comments on steps [2]–[5] follow.

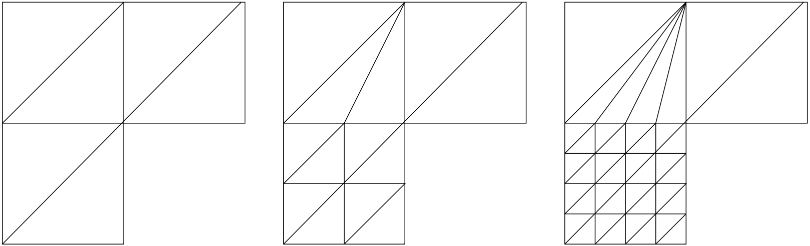
Triangulation — Quasi-Uniform



Automatic triangulation of Ω can be based on successive refinement of an initial coarse (user-defined) triangulation; we could refine each triangle by connecting the midpoints of each side (as above).

This process leads to a uniform or *quasi-uniform* mesh — all triangles will have essentially the same size in all parts of Ω. (Above: since we started with a uniform mesh, it stays uniform.)

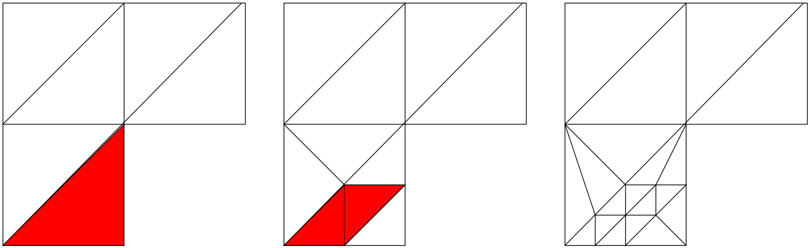
Triangulation — Local Refinement



It is often desirable to construct triangulations where the size of the triangles varies considerably in different parts of the domain. We need smaller triangles where the solution varies quickly. If we know that the right-hand-side of the equation is particularly large in a region, that's probably where we need the triangles...

Note that with the above simple scheme, we generate long narrow triangles — that may lead to problems...

Triangulation — Adaptive



First we compute the solution on the coarse grid **and** error estimates for each triangle. In the regions where the error is too large, we refine — and keep going until we reach some tolerance.

Computation

The next step is to compute the element stiffness matrices, *i.e.* the inner products

$$a|_t(\phi_i, \phi_j), \;\; i, j = 1, 2, \ldots, N_{\text{nodes}}, \quad \forall t \in T_h$$

Most of these are zeros — only overlapping basis functions contribute.

Then we compute the element load vectors

$$(f, \phi_i)_t, \;\; i = 1, 2, \ldots, N_{\text{nodes}}, \quad \forall t \in T_h$$

Assembly

Once we have the element stiffness matrices and load vectors, we are ready to assemble the contributions to the global element stiffness matrix and load vector...

This (and the steps above) require very good bookkeeping — we need to number the nodes in an intelligent way (so we get a good — banded — structure of the stiffness matrix), and also keep track of all the node-triples which define the triangles.

We also have to consider storage of A — most of the entries will be zero (the matrix is **sparse**). Even for moderately sized problems, storing a sparse matrix in "dense" / "full" format requires lots of memory.

Solution

Finally we have to solve the linear system

$$A\tilde{\xi} = \tilde{\mathbf{b}}$$

where A is a large, sparse matrix.

This is a science in itself and some techniques are discussed in Math 543.

In summary we notice that it is quite easy to write down the FEM formulation of a problem, but we have to invest quite a bit of work to get to the final solution. — That's a good argument for using some off-the-shelf software package.