Numerical Solutions to Differential Equations
Lecture Notes #8 — Example: The Lorenz System

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Outline

1. Intermission: Example Time!
   - Today's Problem: The Lorenz System

2. The Lorenz System
   - Introduction
   - Analysis... and Simulations

Intermission...

We have a collection of quite powerful methods for solving **Initial Value Problems** for **Ordinary Differential Equations**

$$y'(t) = f(t, y), \quad y(0) = y_0, \quad t \in [0, T]$$

We have looked (in some detail) at Taylor Series methods, Explicit Runge-Kutta methods, and Linear Multistep methods (both of Adams-type and BDF formulas).

There are still some important issues and topics to cover in regards to initial value problems, *e.g* **predictor-corrector** methods and **"stiffness"** (multi-scale behavior).

Then, there's Boundary Value Problems, and solutions to Delay Differential Equation and Partial Differential Equations...

Intermission... EXAMPLES!!!

The last couple of lectures seemed to get "a bit" abstract, so it's time for an **EXAMPLE!!!**

We're going to use Runge's 4-stage and Adams-Bashforth's 4th order methods to study the **Lorenz system** (Edward Lorenz, 1963), a model of two-dimensional convection in a horizontal layer of fluid heated from below. It is a simple model for Rayleigh-Bénard Convection.
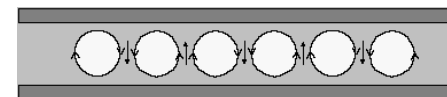
Figure: "Convection rolls" in a fluid layer confined between two thermally conducting plates, heated from below.

Rayleigh-Bénard Convection, I

This type of convection pattern occurs in a relatively shallow layer — this could mean a layer of fluid 1 millimeter thick in a petri dish, or the first 2 kilometers of the Earth's atmosphere.

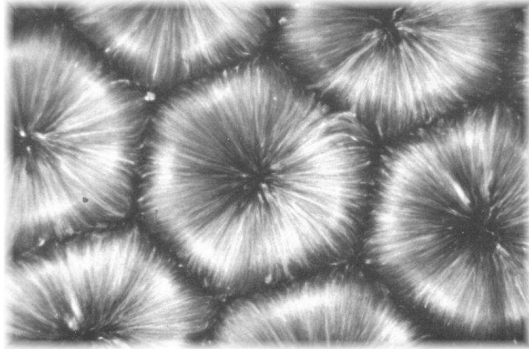Under the right conditions, convection cells will take the shape of hexagons.

Image boldly "borrowed" from NOAA

Rayleigh-Bénard Convection, II

Why don't we see hexagon-shaped clouds in the sky? Take a look at the picture, and notice the small glitch in the pattern. The glitch is due to a tiny defect in the medium — The pattern is very sensitive to the bottom surface. Think about our earth — its surface has millions of dents and bumps in the form of mountains, valleys, canyons, and more. All of these surface features affect the convection patterns in the atmosphere.
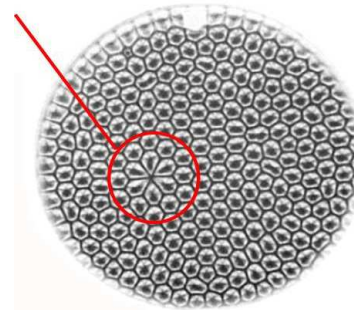

Image boldly "borrowed" from NOAA

The Lorenz System

Lorenz' simplified model:

$$
\begin{aligned}
x' &= \sigma(y - x) \\
y' &= rx - y - zx \\
z' &= -bz + xy
\end{aligned}
$$

Where x represents velocity and y, z the temperature of the fluid. r, σ, and b are positive parameters determined by the heating, the physical properties of the fluid, and the height of the layer.

The above is a convenient lie... x, y, z are the first coefficients of a Fourier expansion of the velocity and temperature fields. (Too much information).

r is the **Rayleigh number**, σ the **Prandtl number**, and b related to the frequency of oscillations in the field.

The Lorenz System, What do the parameters mean???

The Rayleigh number (r) is a dimensionless measure of the temperature difference across the layer, representing the ratio of the destabilizing buoyancy forces to the stabilizing forces due to molecular diffusion of momentum and heat.

The Prandtl number (σ) is the ratio of the coefficients of kinematic viscosity and thermal diffusion of the fluid.

The Lorenz System: Taking a closer look

The non-linear system

$$
\begin{aligned}
x' &= \sigma(y - x) \\
y' &= rx - y - zx \\
z' &= -bz + xy
\end{aligned}
$$

has an equilibrium point $x = y = z = 0$ for all values of r, and

$$
x = y = \pm\sqrt{b(r-1)}, \quad z = r - 1, \qquad r > 1
$$

Stability Analysis of the Equilibria

If we linearize (Taylor expansion) the system around $x = y = z = 0$:

$$
\begin{aligned}
x' &= \sigma(y - x) \\
y' &= rx - y \\
z' &= -bz
\end{aligned}
$$

we get $z(t) = z_0 e^{-bt}$ and $x', y' \propto e^{\lambda t}$ where

$$
0 = \begin{vmatrix} \lambda + \sigma & -\sigma \\ -r & \lambda + 1 \end{vmatrix} = \lambda^2 + (\sigma + 1)\lambda + \sigma(1 - r)
$$

which gives the eigenvalues

$$
\lambda_{1,2} = -\frac{1}{2}(\sigma + 1) \pm \frac{1}{2}\sqrt{(\sigma + 1)^2 + 4\sigma(r - 1)}
$$

This shows that $(x, y, z) = (0, 0, 0)$ is a stable solution for $r < 1$ and unstable for $r > 1$.

Stability Analysis of the Equilibria, II

In order to analyze the remaining equilibrium points ($r > 1$) we make a change of variables (moving the points to $(0, 0, 0)$ in the new coordinates):

$$
x_* = x \mp \sqrt{b(r-1)}, \quad y_* = y \mp \sqrt{b(r-1)}, \quad z_* = z - (r - 1)
$$

Linearizing around $(x_*, y_*, z_*) = (0, 0, 0)$ gives

$$
\begin{aligned}
x_*' &= \sigma(y_* - x_*) \\
y_*' &= x_* - y_* \mp \sqrt{b(r-1)}z_* \\
z_*' &= \pm\sqrt{b(r-1)}(x_* + y_*) - bz_*
\end{aligned}
$$

Hence $x_*, y_*, z_* \propto e^{\lambda t}$ where the eigenvalues are given by...

Stability Analysis of the Equilibria, III ($r > 1$)

$$
\begin{aligned}
0 &= \begin{vmatrix} \lambda + \sigma & -\sigma & 0 \\ -1 & \lambda + 1 & \pm\sqrt{b(r-1)} \\ \mp\sqrt{b(r-1)} & \mp\sqrt{b(r-1)} & \lambda + b \end{vmatrix} \\
&= \lambda^3 + (\sigma + b + 1)\lambda^2 + b(\sigma + r)\lambda + 2b\sigma(r - 1) \\
&= p(\lambda)
\end{aligned}
$$

The roots of $p(\lambda)$ determine the stability...

Since all the coefficients $\{1, (\sigma + b + 1), b(\sigma + r), 2b\sigma(r - 1)\}$ are positive, $p(\lambda) > 0, \forall \lambda \in \mathbb{R}^+$. Hence **instability** occurs only when there are two complex conjugate roots of $p(\lambda)$, $\lambda_{1,2} = \alpha \pm \beta i$, and $\alpha > 0$.

Stability Analysis of the Equilibria, IV

For $r = 1$ the roots are $\lambda_{1,2,3} = \{0, -b, -(\sigma + 1)\}$ (marginally stable).

The first root (for r slightly larger than 1)

$$\lambda_1(r) \sim \frac{-2\sigma(r-1)}{\sigma + 1} < 0, \quad r = 1 + \epsilon$$

After lots of (omitted) algebra, we can summarize the stability:
The points $(x, y, z) = (\pm\sqrt{b(r-1)}, \pm\sqrt{b(r-1)}, r-1)$ are stable if and only if either

$$(a) \quad \sigma < b + 1 \text{ and } r > 1 \quad \text{or}$$
$$(b) \quad \sigma > b + 1 \text{ and } 1 < r < r_c$$

where $r_c = \dfrac{\sigma(\sigma + b + 3)}{\sigma - b - 1}$.

Moving Forward   (Drazin, Nonlinear Systems [1992])

A lot more can be said about the stability analysis for the Lorenz system... We are going to fix $\sigma = 10$ and $b = 8/3$ and study the behavior of the system as r grows from 0 to ∞:

r interval	Stability of Lorenz
(0, 1)	1 Equilibrium point (EP)
(1, 24.06)	2 EPs
(24.06, 24.74)	2 EPs + "Strange Attractor"
(24.74, 30.1)	"Strange Attractor"
(30.1, 99.52)	"Strange Attractor" or Limit Cycle
(99.52, 100.79)	Limit Cycle
(100.79, 145)	"Strange Attractor" or Limit Cycle
(145, 166)	Limit Cycle
(166, 214.36)	"Strange Attractor" or Limit Cycle
(214.36, ∞)	Limit Cycle

The Numerical Simulator   (Matlab Code, I)

```
function v = L(sigma,r,b,x)
   v = zeros(size(x));
   v(1) = sigma*( -x(1)+x(2) );
   v(2) = r*x(1)-x(2)-x(3)*x(1);
   v(3) = -b*x(3)+x(1)*x(2);
return

AB4 = [55 -59 37 -9]/24;
x0 = [0 1 0]';
s = 10;      b = 8/3;
h = 0.001;   Tmax = 30;
T = 0:h:Tmax;
```

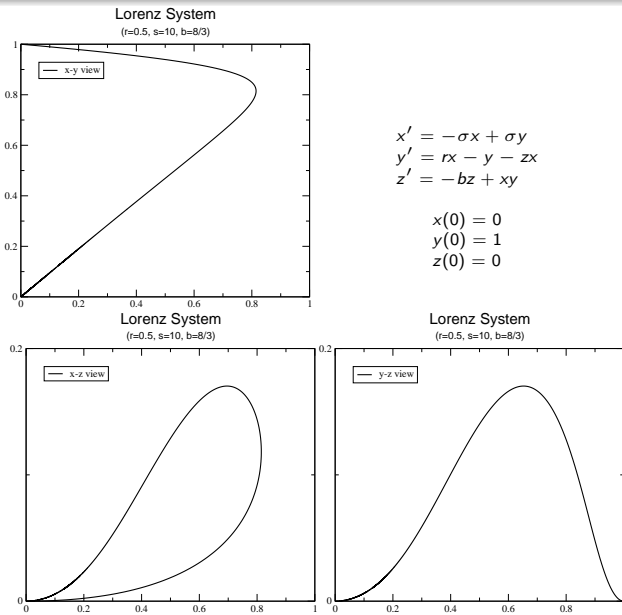## The Numerical Simulator   (Matlab Code, II)

```
for r = [0.5 2.5 10 24 28 55 100 101 150 170 220]
   y = zeros(3,length(T));
   y(:,1) = x0;
   for ix = 1:(length(T)-1)
      k1 = L(s,r,b,y(:,ix));
      k2 = L(s,r,b,y(:,ix)+1/2*h*k1);
      k3 = L(s,r,b,y(:,ix)+1/2*h*k2);
      k4 = L(s,r,b,y(:,ix)+h*k3);
      y(:,ix+1) = y(:,ix) + h/6*(k1 + 2*k2 + 2*k3 + k4);
      f_history(:,ix) = k1;
   end
   ix = 4; f_history(:,ix) = L(s,r,b,y(:,ix));
   f_history = fliplr(f_history);
   for ix = 4:(length(T)-1)
      y(:,ix+1) = y(:,ix) + h * f_history * AB4';
      f_history = [ L(s,r,b,y(:,ix+1)) f_history(:,1:3) ];
   end
   % (Plotting Omitted for Space)
end
```
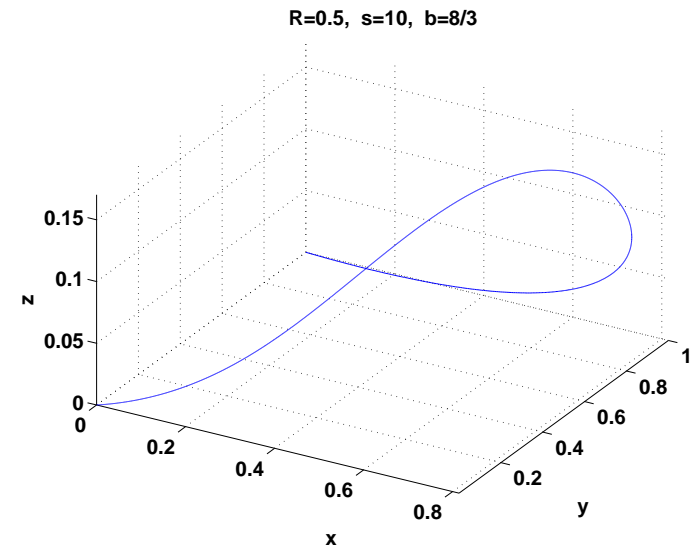
Lorenz System
(r=0.5, s=10, b=8/3)

x-y view

$$x' = -\sigma x + \sigma y$$
$$y' = rx - y - zx$$
$$z' = -bz + xy$$

$$x(0) = 0$$
$$y(0) = 1$$
$$z(0) = 0$$

Lorenz System
(r=0.5, s=10, b=8/3)

x-z view

Lorenz System
(r=0.5, s=10, b=8/3)

y-z view

R=0.5, s=10, b=8/3

z

x

y

Lorenz System
(r=2.5, s=10, b=8/3)

x-y view

$$x' = -\sigma x + \sigma y$$
$$y' = rx - y - zx$$
$$z' = -bz + xy$$

$$x(0) = 0$$
$$y(0) = 1$$
$$z(0) = 0$$

Lorenz System
(r=2.5, s=10, b=8/3)

x-z view

Lorenz System
(r=2.5, s=10, b=8/3)

y-z view

R=2.5, s=10, b=8/3

z

x

y

Lorenz System
(r=10, s=10, b=8/3)

x-y view

$$x' = -\sigma x + \sigma y$$
$$y' = rx - y - zx$$
$$z' = -bz + xy$$

$$x(0) = 0$$
$$y(0) = 1$$
$$z(0) = 0$$

Lorenz System
(r=10, s=10, b=8/3)

x-z view

Lorenz System
(r=10, s=10, b=8/3)

y-z view
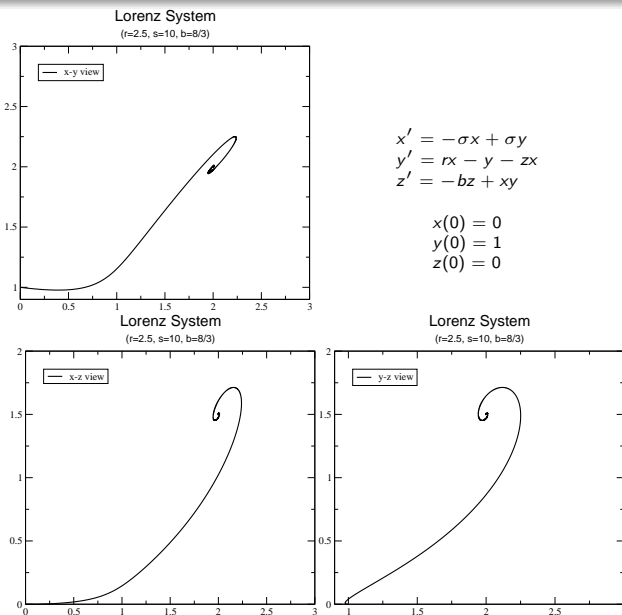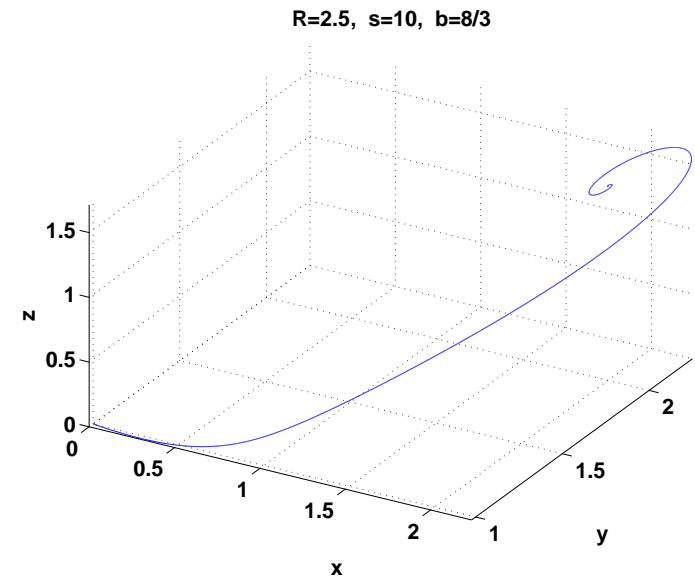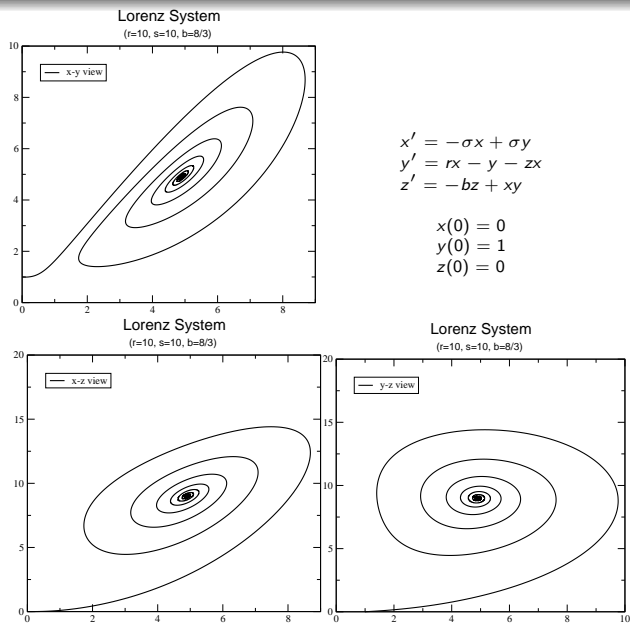
Example: The Lorenz System   — (21/40)

R=10.0, s=10, b=8/3

z

x

y

Example: The Lorenz System   — (22/40)

Lorenz System
(r=24, s=10, b=8/3)

x-y view

$$x' = -\sigma x + \sigma y$$
$$y' = rx - y - zx$$
$$z' = -bz + xy$$

$$x(0) = 0$$
$$y(0) = 1$$
$$z(0) = 0$$

Lorenz System
(r=24, s=10, b=8/3)

x-z view

Lorenz System
(r=24, s=10, b=8/3)

y-z view
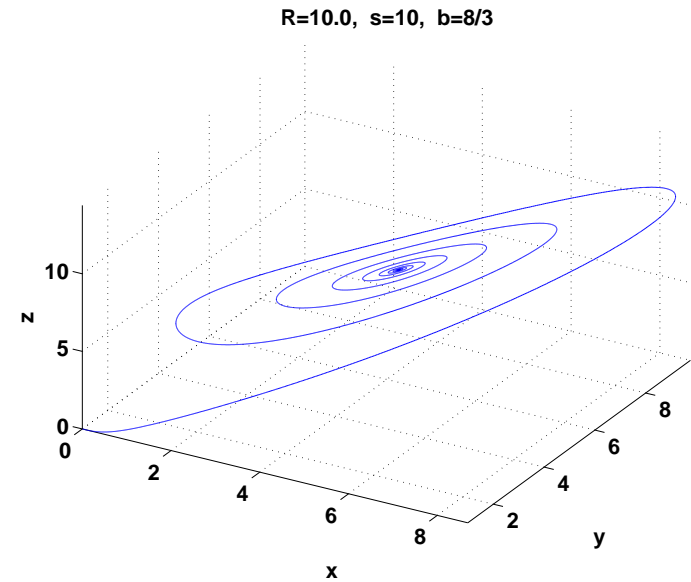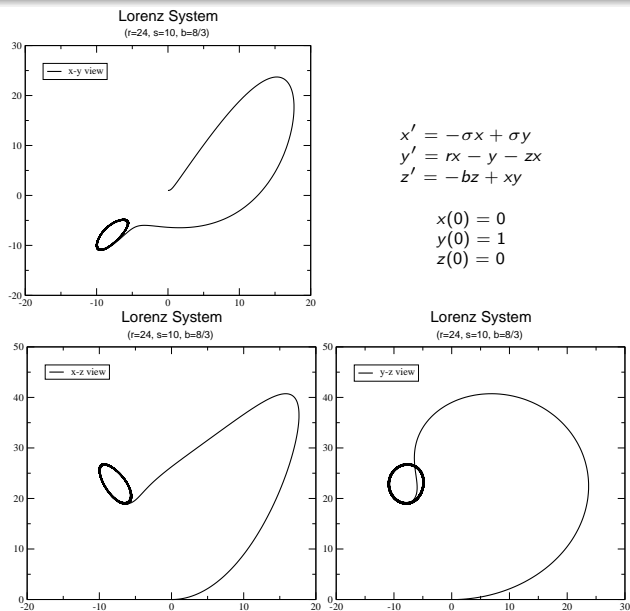
Example: The Lorenz System   — (23/40)

R=24.0, s=10, b=8/3

z

x

y
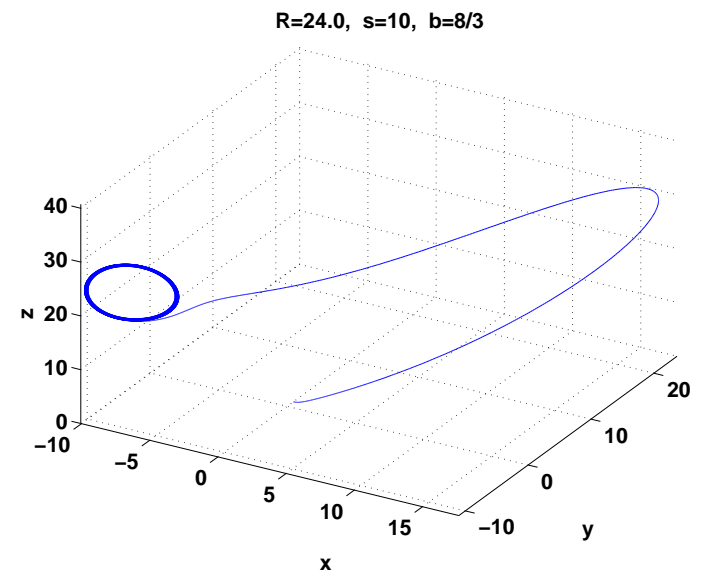
Example: The Lorenz System   — (24/40)
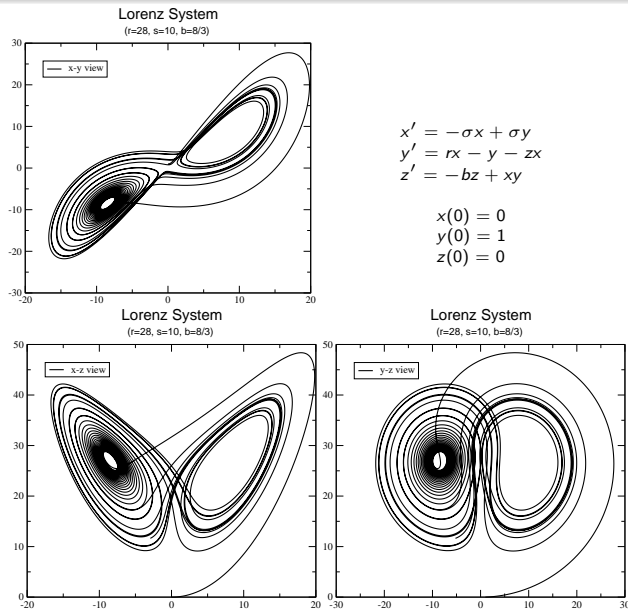
Lorenz System
(r=28, s=10, b=8/3)

— x-y view

$$x' = -\sigma x + \sigma y$$
$$y' = rx - y - zx$$
$$z' = -bz + xy$$

$$x(0) = 0$$
$$y(0) = 1$$
$$z(0) = 0$$

Lorenz System
(r=28, s=10, b=8/3)

— x-z view

Lorenz System
(r=28, s=10, b=8/3)

— y-z view

R=28.0, s=10, b=8/3

Lorenz System
(r=55, s=10, b=8/3)

— x-y view

$$x' = -\sigma x + \sigma y$$
$$y' = rx - y - zx$$
$$z' = -bz + xy$$

$$x(0) = 0$$
$$y(0) = 1$$
$$z(0) = 0$$

Lorenz System
(r=55, s=10, b=8/3)

— x-z view

Lorenz System
(r=55, s=10, b=8/3)

— y-z view

R=55.0, s=10, b=8/3

$$x' = -\sigma x + \sigma y$$
$$y' = rx - y - zx$$
$$z' = -bz + xy$$

$$x(0) = 0$$
$$y(0) = 1$$
$$z(0) = 0$$
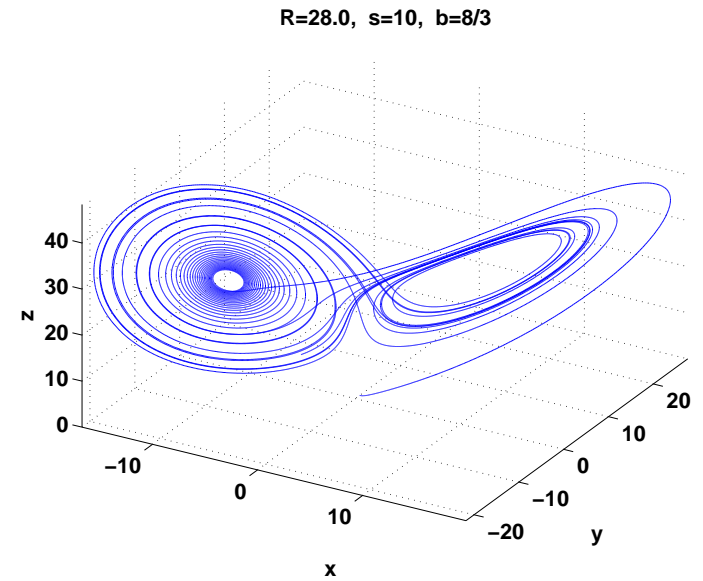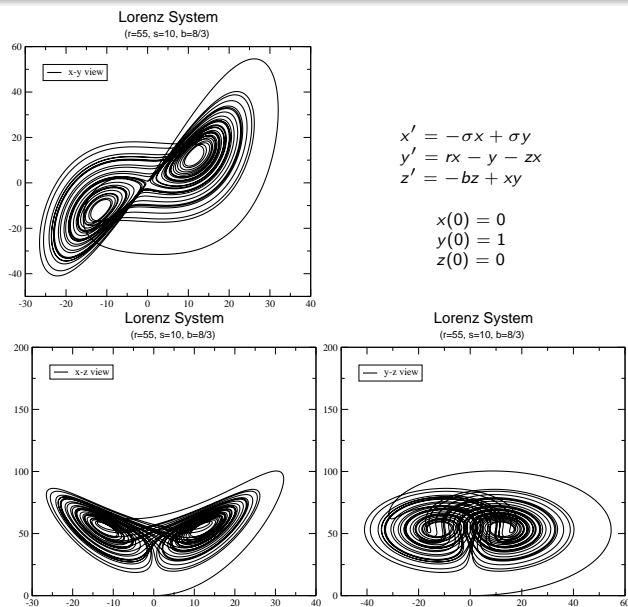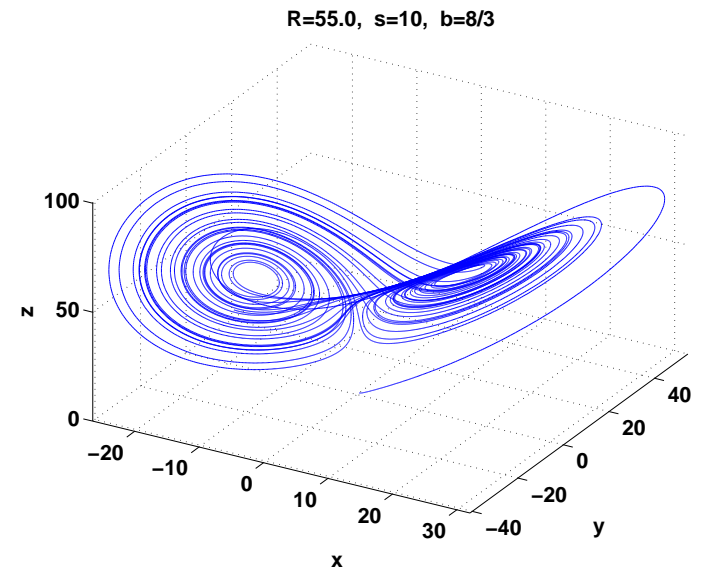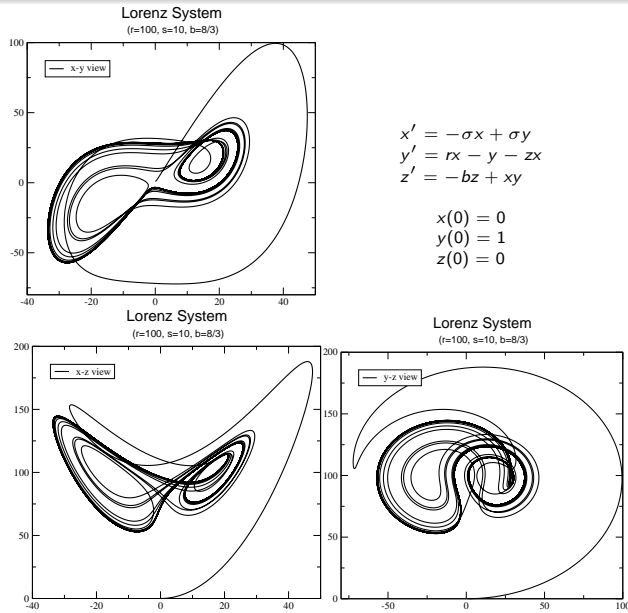
Example: The Lorenz System   — (29/40)

R=100.0, s=10, b=8/3
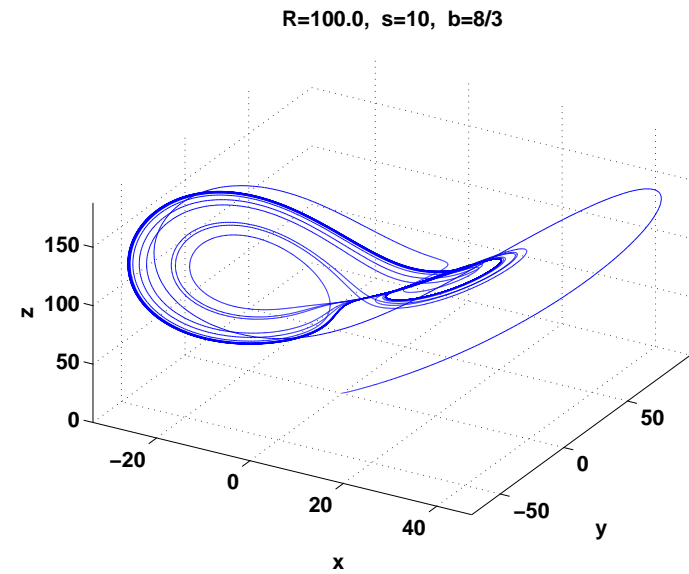
Example: The Lorenz System   — (30/40)

$$x' = -\sigma x + \sigma y$$
$$y' = rx - y - zx$$
$$z' = -bz + xy$$

$$x(0) = 0$$
$$y(0) = 1$$
$$z(0) = 0$$
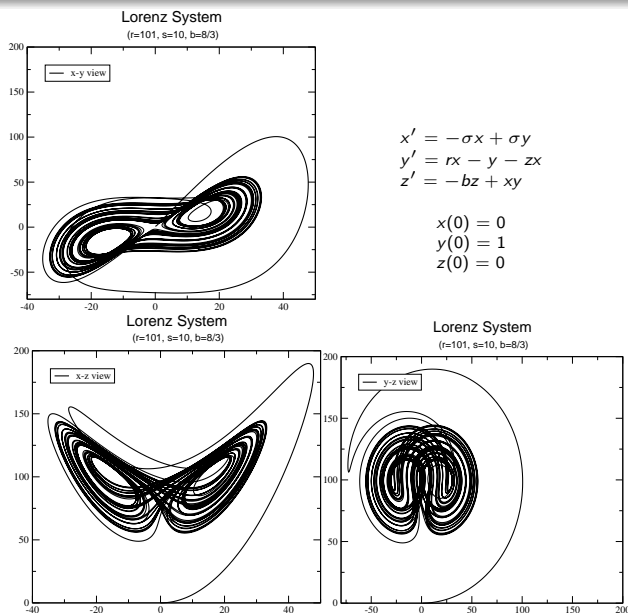
Example: The Lorenz System   — (31/40)

R=101.0, s=10, b=8/3

Example: The Lorenz System   — (32/40)

Lorenz System
(r=150, s=10, b=8/3)

x-y view

$$x' = -\sigma x + \sigma y$$
$$y' = rx - y - zx$$
$$z' = -bz + xy$$

$$x(0) = 0$$
$$y(0) = 1$$
$$z(0) = 0$$

Lorenz System
(r=150, s=10, b=8/3)

x-z view

Lorenz System
(r=150, s=10, b=8/3)

y-z view
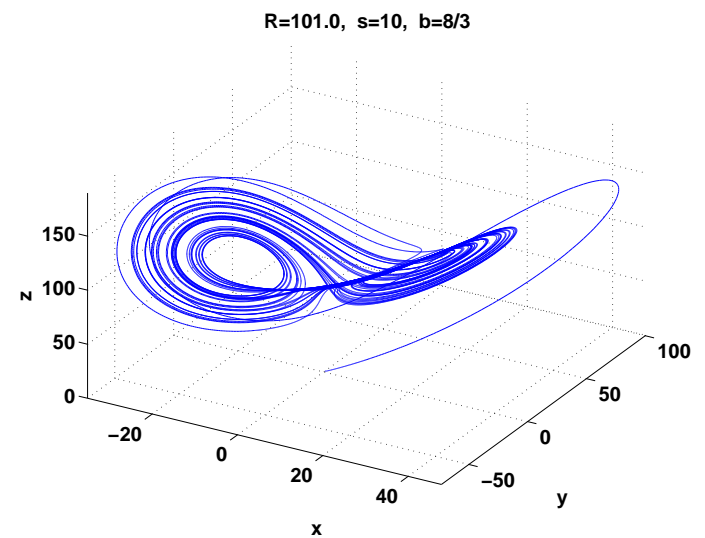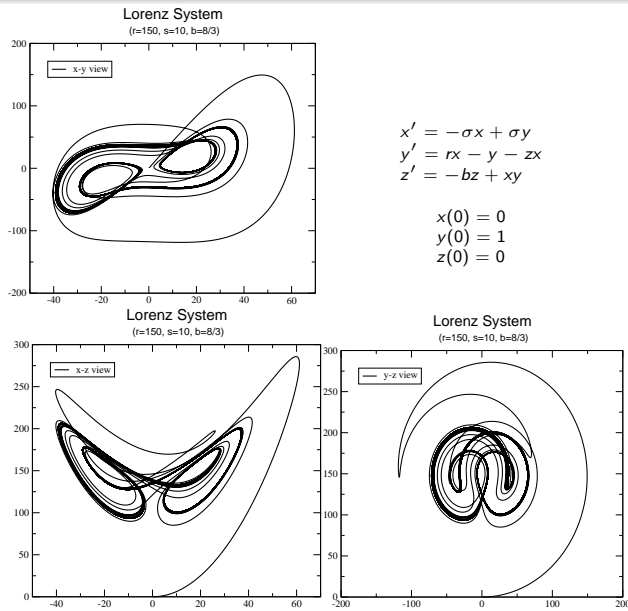
Example: The Lorenz System   — (33/40)

R=150.0, s=10, b=8/3

Example: The Lorenz System   — (34/40)

Lorenz System
(r=170, s=10, b=8/3)

x-y view

$$x' = -\sigma x + \sigma y$$
$$y' = rx - y - zx$$
$$z' = -bz + xy$$

$$x(0) = 0$$
$$y(0) = 1$$
$$z(0) = 0$$

Lorenz System
(r=170, s=10, b=8/3)

x-z view

Lorenz System
(r=170, s=10, b=8/3)

y-z view
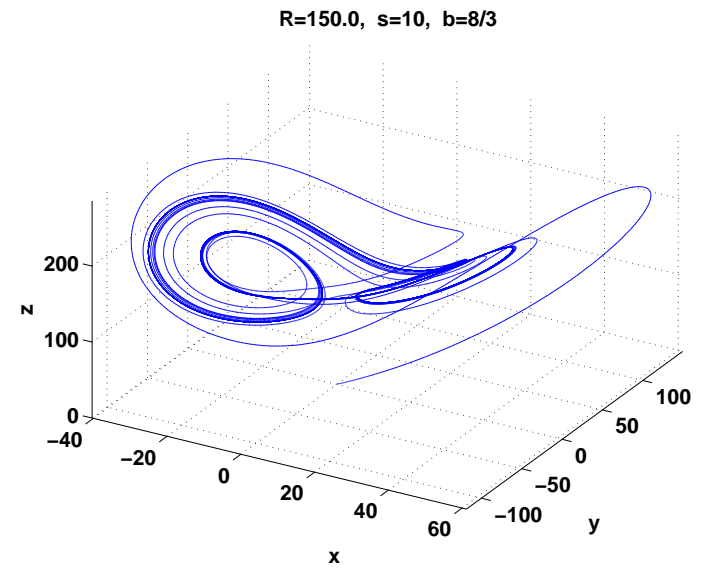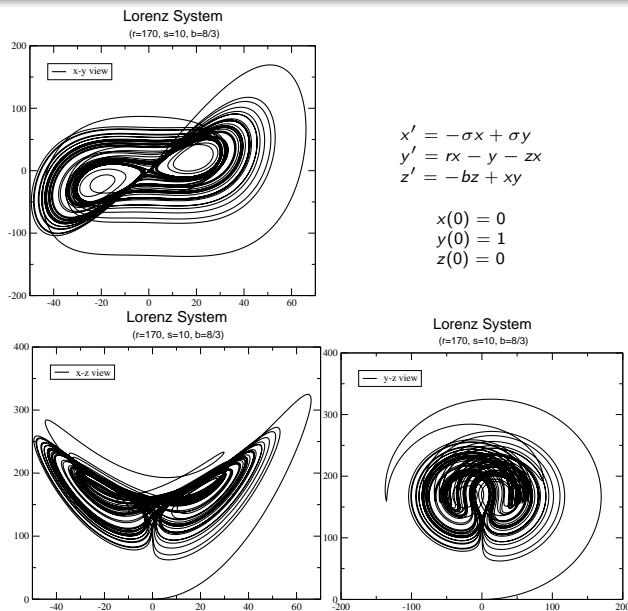
Example: The Lorenz System   — (35/40)

R=170.0, s=10, b=8/3

Example: The Lorenz System   — (36/40)

Lorenz System
(r=220, s=10, b=8/3)

x-y view

$$x' = -\sigma x + \sigma y$$
$$y' = rx - y - zx$$
$$z' = -bz + xy$$

$$x(0) = 0$$
$$y(0) = 1$$
$$z(0) = 0$$

Lorenz System
(r=220, s=10, b=8/3)

x-z view

Lorenz System
(r=220, s=10, b=8/3)

y-z view

R=220.0, s=10, b=8/3