Numerical Solutions to Differential Equations

Lecture Notes #18

Boundary Value Problems – Introduction

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Outline

Boundary Value Problems

We will now consider Ordinary Differential Equations where conditions (constraints) are specified at more than one point (typically at the start and end points) of the independent variable.

The ODE itself can be **linear** or **non-linear**.

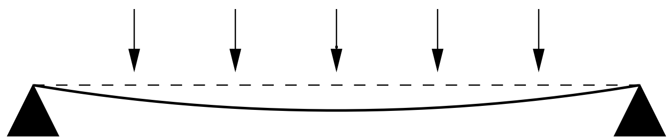
The boundary conditions (BCs) can also be **linear** or **non-linear**.

Further, the boundary conditions can be **separated** or **mixed**.

A Boundary Value Problem (BVP) is considered **non-homogeneous** if the ODE or any of the BCs contain at least one non-zero term which is independent of the function we are solving for (or its derivatives).

Applications which give rise to Boundary Value Problems

The favorite example, which appeared in every reference I looked at, is the **deflection of a beam subject to load:** (books on a bookshelf?)



Clearly the deflection is **zero** at the endpoints

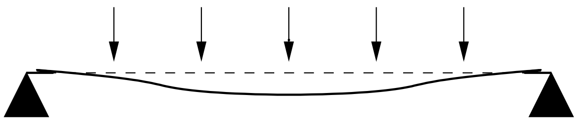
$$y(0) = y(L) = 0,$$

since the beam is resting on supports.

Deflection of a Beam Subject to Load

If it is a bookshelf, the end-points are usually clamped in, which means there is no slope at the end-points:

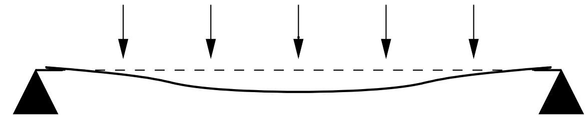
$$y(0) = y(L) = 0, \quad y'(0) = y'(L) = 0.$$



More twists to the problem:

- The load can be either uniform, or distributed.
- The shelf (beam) itself can have varying geometry.

Deflection of a Beam Subject to Load — Questions



Engineering Questions of Interest:

- Maximal load — deflection is too large?
- Maximal load — permanent shape change (some materials)?
- Maximal load — catastrophic shape change (breakage)?

Deflection of a Beam Subject to Load: Equation

Transverse deflection of a beam $w(x)$ subject to distributed load, $p(x)$

$$\frac{d^2}{dx^2}\left[E(x)I(x)\frac{d^2w(x)}{dx^2}\right] = p(x)$$

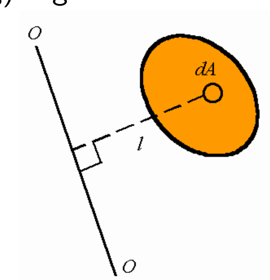
$E(x)$ — **Young's Modulus**
Young's Modulus is the **stress** of a material divided by its **strain**. That is how much the material **yields** for each unit of force loading it. Put another way, it is a measure of the **strength** of a material, and is commonly used to measure the strength of metals and other materials used in for instance aircraft and building materials.

$I(x)$ **area moment of inertia*** **of the beam.**

The Area Moment of Inertia

From http://em-ntserver.unl.edu/NEGAHBAN/EM223/note18/note18.htm:

The **area moment of inertia** is the second moment of area around a given axis. For example, given the axis O-O and the shaded area shown, one calculates the second moment of the area by adding (integrating) together $\ell^2 dA$ for all the elements



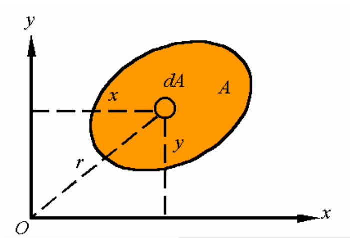
of area dA in the shaded area.

The area moment of inertia, denoted by I, can, therefore, be calculated from

$$I = \int_A \ell(A)^2 dA.$$

If we have a rectangular coordinate system as shown, one can define the area moment of inertial around the x-axis, denoted by I_x, and the area moment of inertia about the y-axis, denoted by I_y. These are given by

$$I_x = \int_A y^2 dA, \quad I_y = \int_A x^2 dA.$$



The polar area moment of inertia, denoted by J_O, is the area moment of inertia about the z-axis given by

$$J_O = \int_A r^2 dA.$$

Note that since $r^2 = x^2 + y^2$ one has the relation

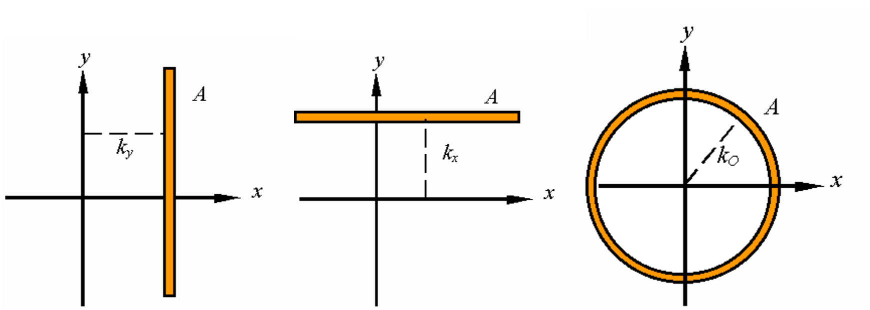
$$J_O = I_x + I_y.$$

The radius of rotation is the distance k away from the axis that all the area can be concentrated to result in the same moment of inertia. That is,

$$I = k^2 A.$$

For a given area, one can define the radius of rotation around the x-axis, denoted by k_x, the radius of rotation around the y-axis, denoted by k_y, and the radius of rotation around the z-axis, denoted by k_O. These are calculated from the relations

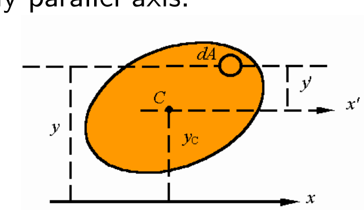
$$k_x^2 = \frac{I_x}{A}, \quad k_y^2 = \frac{I_y}{A}, \quad k_O^2 = \frac{J_O}{A}$$



From the relation $J_O = I_x + I_y$ it is easy to show that

$$k_x^2 + k_y^2 = k_O^2.$$

The **parallel axis theorem** is a relation between the moment of inertia about an axis passing through the centroid and the moment of inertia about any parallel axis.



Note that from the picture we have

$$\begin{aligned} I_x &= \int_a y^2 dA = \int_A (y_c + y')^2 dA = \int_a y_c^2 dA + \int_A y'^2 dA + 2\int_A y_c y' \, dA \\ &= y_c^2 A + I_{x'} + 2y_c \int_A y' \, dA \end{aligned}$$

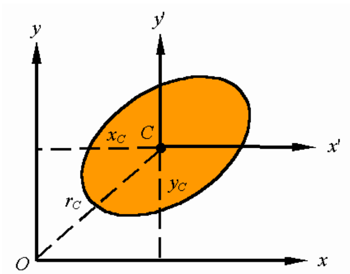
The Area Moment of Inertia

Since $\frac{1}{A}\int_a y' dA$ gives the distance of the centroid above the x'-axis, and since the this distance is zero, one must conclude that the integral in the last term is zero so that the parallel axis theorem states that

$$I_x = I_{x'} + Ay_c^2$$

where x' must pass through the centroid of the area. In this same way, one can show that

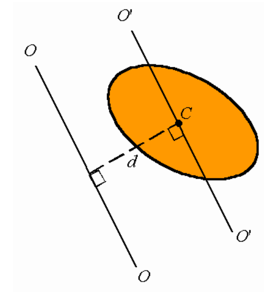
$$I_y = I_{y'} + Ax_c^2, \quad J_O = J_{O'} + AR_c^2 = J_{O'} + A(x_c^2 + y_c^2)$$



The Area Moment of Inertia

In general, one can use the parallel axis theorem for any two parallel axes as long as one passes through the centroid. As shown in the picture, this is written as

$$I = \bar{I} + Ad^2$$

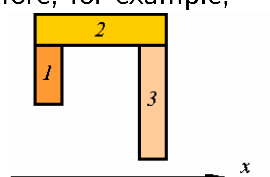


where $\bar{I}$ is the moment of inertia about the axis O'-O' passing through the centroid, I is the moment of inertia about the axis O-O, and d is the perpendicular distance between the two parallel axes.

The Area Moment of Inertia

The moment of inertia of composite bodies can be calculated by adding together the moment of inertia of each of its sections. The only thing to remember is that all moments of inertia must be evaluated bout the same axis. Therefore, for example,

$$I_x = \sum_{j=1}^{n} I_x^{(j)}$$



To calculate the area moment of inertia of the composite body constructed of the three segments shown, one evaluates the moment of inertial of each part about the x-axis and adds the three together.

That's definitely more detail than we need for this class!!! But it was a very nice explanation...

(Area) Moments of Inertia

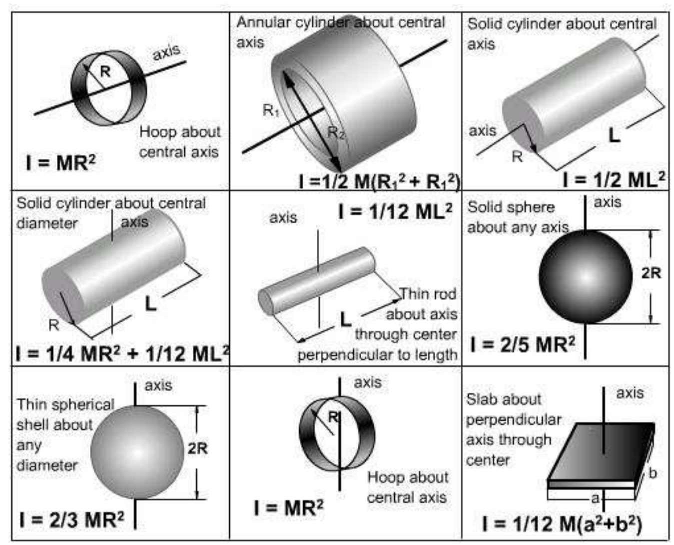


Figure: Source `tutorvista.com`, 3/22/2013.

Back to the Beam Equation

We had

$$\frac{d^2}{dx^2}\left[E(x)I(x)\frac{d^2w(x)}{dx^2}\right] = p(x).$$

If/When the beam is uniform, then $E(x) = E$, and $I(x) = I$ are constants, and the equation simplifies to

$$\frac{d^4w(x)}{dx^4} = \frac{p(x)}{EI}.$$

Boundary Conditions for the Beam Equation

For a fixed beam (shelves) the deflection and slope are zero at the end-points

$$w(0) = w(L) = \frac{dw(0)}{dx} = \frac{dw(L)}{dx} = 0.$$

For a simply supported beam, the deflection and bending are zero at the end-points

$$w(0) = w(L) = \frac{dw^2(0)}{dx} = \frac{dw^2(L)}{dx} = 0.$$

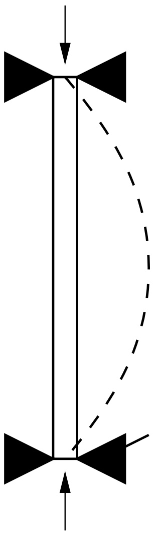
Alternative ways to Abuse Beams...

A pin-ended uniform column of length L is subject to an axial load P as shown in the figure to the right. The transverse deflection of the column $w(x)$ is described by the following equation

$$EI\frac{d^2w(x)}{dx^2} + Pw(x) = 0.$$

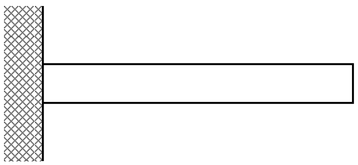
Since the column is pin-connected at the ends, the transverse deflection is zero

$$w(0) = w(L) = 0.$$



Cooling Fins

There's quite a bit metal cooling fins sitting on top of a modern CPU!



The fin extends from the heat source (left) which has temperature T_0; the surrounding air temperature is T_∞. The temperature in the cooling fin (at steady-state) is described by the following ODE

$$kA\frac{d^2T}{dx^2} - hP(T - T_\infty) = 0.$$

We're assuming heat-flow only in the x-direction; k is the thermal conductivity, A the area of the cross-section, h the convection heat transfer coefficient, P the perimeter of the fin.

Cooling Fins — Boundary Conditions

At the heat source the temperature in the fin equals the temperature of the source

$$T(0) = T_0.$$

At the "free" end of the fin, the convection loss of the fin must equal the heat transfer by conduction (no local build-up of heat)

$$-k\frac{dT(L)}{dx} = h(T(L) - T_\infty).$$

∃ Useful Applications == TRUE

Now that we have seen that if we want to build bridges and buildings, or cool computer chips, we must solve some boundary value problems the natural question is **how???**

There are several approaches to solving BVPs:

- **Shooting methods** — we convert the BVP into a sequence of Initial Value Problems, thus re-using all the tools we talked about previously!
- **Finite difference methods** — the ODE is converted into a set of simultaneous algebraic equations (directly applicable to higher order derivatives).
- **Finite element methods (The Rayleigh-Ritz method)** — A completely different point of view! Can be seen as the selection from the set of all sufficiently differentiable functions satisfying the boundary conditions, the function which minimizes a certain integral.

Shooting Methods

Applicable to: Linear and non-linear BVPs.

Easy to implement.

Fast convergence, when it works.

However, no guarantee of convergence!!!

Shooting Methods — The Procedure

[1] The unspecified initial conditions of the differential equation are **guessed** so that the problem can be solved as an initial value problem.

[2] The variational equations denoting the sensitivity of the dependent variables with respect to the guessed initial conditions are **derived**.

[3] The differential equation and the variational equations are **integrated** along the x-direction as a set of simultaneous initial value problems.

[4] The result of the sensitivities found in step [3] are used to **correct** the guessed initial conditions.

[5] With the new (corrected) initial conditions, **repeat** steps [2], [3] and [4] **until** the specified second (terminal) boundary conditions are satisfied.

Consider the BVP

$$\frac{d^2y(x)}{dx^2} = f\left(x, y, y'\right), \quad a \le x \le b, \quad y(a) = y_a, \quad y'(b) = y'_b$$

By introducing

$$y_1(x) = y(x), \quad y_2(x) = \frac{dy_1(x)}{dx} = \frac{dy(x)}{dx}$$

we write the problem as a system of two first-order ODEs

$$\begin{aligned} &y_1'(x) = y_2(x) \\ &y_2'(x) = f(x, y_1, y_2) \\ &y_1(a) = y_a, \quad y_2(b) = y'_b \end{aligned}$$

To solve this as an initial value problem, we guess $y_2(a) = Y$ and solve:

$$\begin{array}{rcl} y_1'(x) & = & y_2(x) \\ y_2'(x) & = & f(x, y_1, y_2) \\ \hline y_1(a) & = & y_a \\ y_2(a) & = & Y \end{array}$$

for $x \in [a, b]$.

The final value of y_2 will now depend on the guess Y, *i.e.* $y_2(b) = y_2(b, Y)$.

We define the discrepancy function

$$h(Y) = y_2(b, Y) - y'_b$$

The truncated Taylor series expansion of $h(Y + \Delta Y)$ is

$$h(Y + \Delta Y) = h(Y) + \Delta Y\, h'(Y)$$

We notice that (in the limit $\Delta Y \to 0$)

$$h'(y) = \frac{\partial}{\partial y}\left[y_2(b, Y) - y'_b\right] = \frac{\partial y_2(b, Y)}{\partial Y}$$

Since we are looking for the zero-discrepancy solution, we want

$$0 = h(Y) + \Delta Y\, h'(Y)$$

which gives

$$\Delta Y = -\frac{y_2(b, Y) - y'_b}{\left[\frac{\partial y_2(b,Y)}{\partial Y}\right]}$$

Note: we don't have the quantity $\frac{\partial y_2(b,Y)}{\partial Y}$ (yet).

The new initial guess of the initial condition on $y_2(x)$ becomes:

$$y_2(a) = Y + \Delta Y = Y - \frac{y_2(b, Y) - y'_b}{\left[\frac{\partial y_2(b,Y)}{\partial Y}\right]}$$

The "missing" quantity can be obtained in two ways —

[1] The "theoretical" way... by adding additional ODEs to the system.

[2] The "practical" way... by computing finite differences of the final values $y_2(b, Y^{(1)})$, $y_2(b, Y^{(2)})$, ...

Often, [2] is usually the way to go; but let's look at approach [1]:

Computing $\frac{\partial y_2(b,Y)}{\partial Y}$ — Variational Approach

Computing the partial derivatives of the ODEs give

$$\frac{\partial}{\partial Y}\left[\frac{dy_1}{dx}\right] = \frac{d}{dx}\left[\frac{\partial y_1}{\partial Y}\right] = \frac{\partial f_1}{\partial y_1}\frac{\partial y_1}{\partial Y} + \frac{\partial f_1}{\partial y_2}\frac{\partial y_2}{\partial Y}$$

$$\frac{\partial}{\partial Y}\left[\frac{dy_2}{dx}\right] = \frac{d}{dx}\left[\frac{\partial y_2}{\partial Y}\right] = \frac{\partial f_2}{\partial y_1}\frac{\partial y_1}{\partial Y} + \frac{\partial f_2}{\partial y_2}\frac{\partial y_2}{\partial Y}$$

We define the **sensitivity functions**

$$g_1 = \frac{\partial y_1}{\partial Y}, \quad g_2 = \frac{\partial y_2}{\partial Y},$$

and get the following system of ODEs

$$\frac{dg_1}{dx} = \frac{\partial f_1}{\partial y_1} g_1 + \frac{\partial f_1}{\partial y_2} g_2$$

$$\frac{dg_2}{dx} = \frac{\partial f_2}{\partial y_1} g_1 + \frac{\partial f_2}{\partial y_2} g_2$$

Computing $\frac{\partial y_2(b,Y)}{\partial Y}$ — Variational Approach, II

The appropriate initial conditions for the system

$$\frac{dg_1}{dx} = \frac{\partial f_1}{\partial y_1} g_1 + \frac{\partial f_1}{\partial y_2} g_2$$

$$\frac{dg_2}{dx} = \frac{\partial f_2}{\partial y_1} g_1 + \frac{\partial f_2}{\partial y_2} g_2$$

are given by

$$g_1(a) = \left.\frac{\partial y_1}{\partial Y}\right|_{x=a} = 0$$

$$g_2(a) = \left.\frac{\partial y_2}{\partial Y}\right|_{x=a} = 1$$

Now, we can solve these ODEs together with the primary system, and the final value $g_2(b) \equiv \frac{\partial y_2(b,Y)}{\partial Y}$.

Computing $\frac{\partial y_2(b,Y)}{\partial Y}$ — Finite Difference Approach

Guess twice: $Y^{(1)}$ and $Y^{(2)}$.

Approximate the derivative

$$\left.\frac{\partial h(Y)}{\partial Y}\right|_{Y^{(2)}} \approx \frac{h(Y^{(2)}) - h(Y^{(1)})}{Y^{(2)} - Y^{(1)}}$$

Iteratively update your initial guess

$$Y^{(k+1)} = Y^{(k)} - \frac{h(Y^{(k)})}{\left[\frac{h(Y^{(k)}) - h(Y^{(k-1)})}{Y^{(k)} - Y^{(k-1)}}\right]}$$

until

$$|h(Y^{(k)})| \leq \mathit{tolerance}$$