Numerical Matrix Analysis

Notes #14 — Conditioning and Stability: Least Squares Problems: Conditioning

Peter Blomgren
⟨blomgren@sdsu.edu⟩

Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

http://terminus.sdsu.edu/

Spring 2024
(Revised: March 21, 2024)

Outline

Student Learning Targets, and Objectives

Target Derivation of the Four Condition Numbers of the Least Squares Problem

Recap: Last Time — Backward Stability of Back-Substitution

We looked at a backward stability proof in gory detail. — The technique is quite straight-forward, albeit somewhat tedious.

— We replace the floating point operators $\oplus$, $\ominus$, $\otimes$, and $\oslash$ with exact mathematical operations + relative error terms, *i.e.* $(x \oplus y) \rightsquigarrow (x+y)(1+\epsilon)$, where $|\epsilon| \leq \varepsilon_{\text{mach}}$.

— Then we **interpret** the error as perturbations on the appropriate part of the problem formulation (so that that computed solution is the exact solution to a nearby problem).

Recap: Last Time

As we used the backward substitution algorithm for the detailed backward **stability** proof; we now turn to least squares problems for a detailed discussion on **conditioning**...

...and we recall that `Accuracy(conditioning,stability)`, so these are all important pieces in the larger *"numerics jigsaw puzzle."*

Rewind (Computational Accuracy)

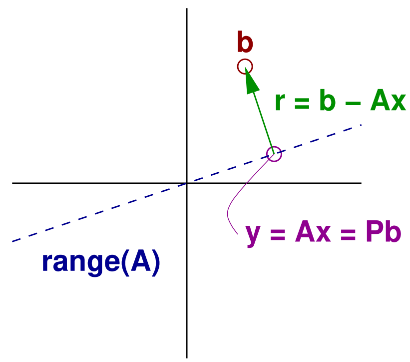
Suppose a backward stable algorithm is applied to solve a problem $f : X \mapsto Y$ with condition number κ in a floating point environment satisfying the floating point representation axiom, and the fundamental axiom of floating point arithmetic.

Then the relative errors satisfy

$$\frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|} = \mathcal{O}(\kappa(x)\varepsilon_{\text{mach}}).$$

Least Squares Problems... 1 of 2

Once again, we return to the least squares problem.



This is easily the most technial lecture of the semester. Grab a barrel of coffee, and enjoy the ride!

We measure everything in the two-norm, and let $\|\cdot\| = \|\cdot\|_2$; formally we are trying to solve

Given $A \in \mathbb{C}^{m\times n}$ of full rank, $m \geq n$, $\vec{b} \in \mathbb{C}^m$, find $\vec{x} \in \mathbb{C}^n$ such that $\|\vec{b} - A\vec{x}\|_2$ is minimized.

Least Squares Problems... 2 of 2

The conditioning of these problems depend on a combination of

(1) The conditioning of square systems of equations
(2) The geometry of orthogonal projections.

The topic is subtle, and has nontrivial implications for the **stability** (and ultimately, the **accuracy**) of least squares algorithms.

From our previous discussion of least squares problem we know

$$\begin{aligned} \vec{x} &= A^\dagger \vec{b}, && \text{where} \quad A^\dagger = (A^*A)^{-1}A^*, \text{ or } R^{-1}Q^*, \text{ or } V\Sigma^{-1}U^* \\ A\vec{x} &= \vec{y}, && \text{where} \quad \vec{y} = P\vec{b}, \quad P = AA^\dagger \end{aligned}$$

P is the **orthogonal projector** onto $\mathrm{range}(A)$, and $A^\dagger \in \mathbb{C}^{n\times m}$ is the **pseudo-inverse** of A. For this theoretical infinite-precision discussion the choice of implementation/expression for the pseudo-inverse does not matter.

Least Squares Problems... Conditioning

Conditioning is the measure of sensitivity of solutions to perturbations in the data.

Our data are

$$A \in \mathbb{C}^{m\times n}, \quad \text{and} \quad \vec{b} \in \mathbb{C}^m,$$

and the solution is either the vector $\vec{x} \in \mathbb{C}^n$, or the vector $\vec{y} = P\vec{b}$ (depending on our point of view / application).

We end up with four combinations of input/output-perturbations:

$\downarrow$ **Input, Output** $\rightarrow$	$\vec{y} \in \mathbb{C}^m$	$\vec{x} \in \mathbb{C}^n$
$\vec{b} \in \mathbb{C}^m$	$\kappa(\vec{b} \mapsto \vec{y})$	$\kappa(\vec{b} \mapsto \vec{x})$
$A \in \mathbb{C}^{m\times n}$	$\kappa(A \mapsto \vec{y})$	$\kappa(A \mapsto \vec{x})$

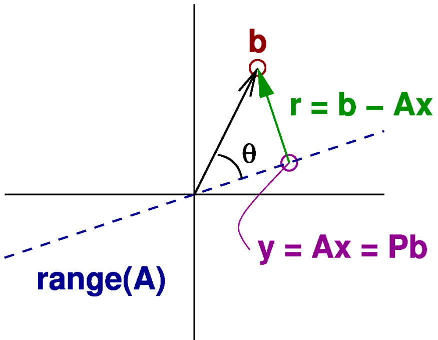
Three Dimensionless Parameters

We are going to express all the condition-numbers using three dimensionless parameters — $\kappa(A)$, θ, and η

$\kappa(A)$ is our old friend the condition number of the matrix A

$$\kappa(A) = \frac{\sigma_1}{\sigma_n}.$$

θ is the angle between $\vec{b}$ and $\vec{y} = A\vec{x} = P\vec{b}$,



Three Dimensionless Parameters

η is a measure of how much $\|\vec{y}\|$ falls short of its maximum value, given $\|A\|$ and $\|\vec{x}\|$: (or how misaligned $(\vec{y}, \vec{x})$ is with $(\vec{u}_1, \vec{v}_1)$. — Implications for "Model Quality")

$$\eta = \frac{\|A\|\,\|\vec{x}\|}{\|\vec{y}\|} = \frac{\|A\|\,\|\vec{x}\|}{\|A\vec{x}\|} = \sigma_1 \frac{\|\vec{x}\|}{\|A\vec{x}\|}.$$

These parameters lie in the ranges

$$\kappa(A) \in [1, \infty), \quad \theta \in \left[0, \frac{\pi}{2}\right], \quad \eta \in [1, \kappa(A)),$$

and

$$\underbrace{\cos(\theta) = \frac{\|\vec{y}\|}{\|\vec{b}\|} \in [0,1],}_{\text{Usually, this is the quantity of interest; not } \theta \text{ itself.}} \qquad \theta = \cos^{-1}\left(\frac{\|\vec{y}\|}{\|\vec{b}\|}\right).$$

Least Squares Problems... Conditioning Theorem

Theorem (Conditioning of Least Squares Problems)

Let $\vec{b} \in \mathbb{C}^m$ and $A \in \mathbb{C}^{m\times n}$ of full rank be given.

The least squares problem, $\min_{\vec{x}\in\mathbb{C}^n} \|\vec{b} - A\vec{x}\|$ has the following 2-norm relative condition numbers describing the sensitivities of $\vec{y}$ and $\vec{x}$ to perturbations in $\vec{b}$ and A:

$\downarrow$ **Input**, **Output** $\rightarrow$	$\vec{y} \in \mathbb{C}^m$	$\vec{x} \in \mathbb{C}^n$
$\vec{b} \in \mathbb{C}^m$	$\dfrac{1}{\cos(\theta)}$	$\dfrac{\kappa(A)}{\eta\cos(\theta)}$
$A \in \mathbb{C}^{m\times n}$	$\dfrac{\kappa(A)}{\cos(\theta)}$	$\kappa(A) + \dfrac{\kappa(A)^2\tan(\theta)}{\eta}$

The results in the first row are exact, being attained for certain perturbations $\delta\vec{b}$, and the results in the second row are upper bounds.

A Note on the Theorem

In the special case $m = n$, the least squares problem reduces to a square non-singular problem, with $\theta = 0$, and the table looks like

$\downarrow$ **Input**, **Output** $\rightarrow$	$\vec{y}$	$\vec{x}$
$\vec{b}$	1	$\dfrac{\kappa(A)}{\eta}$
A	0	$\kappa(A)$

Since A is square + full rank, $\vec{y}$ is already in the range, so no projection is needed; hence the condition number is 0.

Note: Condition numbers less than 1 are rare, and usually indicate that there is no relation between the input and the output.

(Massively) Simplifying the Proof Using the SVD

We have argued (a long, long time ago) that every matrix has a singular value decomposition.

Let $U\Sigma V^* = A$ be the SVD of A. We can use U and V to get two convenient bases in which we prove the theorem. Since 2-norm perturbations are not changed by a unitary change of basis, the **perturbation behavior of A is the same as that of Σ.**

Without loss of generality we can assume that

$$A = \left[\begin{array}{c}\Sigma \\ \hdashline 0\end{array}\right] = \left[\begin{array}{c}A_1 \\ \hdashline 0\end{array}\right] = \left[\begin{array}{cccc}\sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \\ \hdashline 0 & 0 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 0\end{array}\right]$$

Moving Along...

Now with

$$\vec{b} = \left[\begin{array}{c}\vec{b}_1 \\ \hdashline \vec{b}_2\end{array}\right], \quad \vec{b}_1 \in \mathbb{C}^n, \;\; \vec{b}_2 \in \mathbb{C}^{m-n}$$

the projection of $\vec{b}$ onto $\mathrm{range}(A)$ is trivial

$$\vec{y} = P\vec{b} = \left[\begin{array}{c}\vec{b}_1 \\ \hdashline \vec{0}\end{array}\right]$$

Now, $A\vec{x} = \vec{y}$ has the unique solution $\boxed{\vec{x} = A_1^{-1}\vec{b}_1.}$

We note that the orthogonal projector, and the pseudo-inverse of A take the forms

$$P = \left[\begin{array}{c:c} I_{n\times n} & 0 \\ \hdashline 0 & 0\end{array}\right], \qquad A^\dagger = \left[\begin{array}{c:c} A_1^{-1} & 0\end{array}\right].$$

Part#1: Sensitivity of $\vec{y}$ wrt. Perturbations in $\vec{b}$

$\vec{y} = P\vec{b}$ is a linear differentiable map; and the Jacobian is P itself, with $\|P\| = 1$.

For a differentiable map $x \mapsto f(\vec{x})$ the condition number is

$$\kappa(\vec{x}) = \frac{\|J(\vec{x})\|}{\|f(\vec{x})\|/\|\vec{x}\|}.$$

Here we have

$$\kappa(\vec{b} \mapsto \vec{y}) = \frac{\|P\|}{\|\vec{y}\|/\|\vec{b}\|} = \frac{1}{\cos(\theta)}. \quad \square$$

Part#2: Sensitivity of $\vec{x}$ wrt. Perturbations in $\vec{b}$

$\vec{x} = A^\dagger\vec{b}$ is also linear, with Jacobian $J = A^\dagger$, so

$$\kappa(\vec{b} \mapsto \vec{x}) = \frac{\|A^\dagger\|}{\|\vec{x}\|/\|\vec{b}\|} = \|A^\dagger\| \, \frac{\|\vec{b}\|}{\|\vec{y}\|} \, \frac{\|\vec{y}\|}{\|\vec{x}\|} = \|A^\dagger\| \, \frac{1}{\cos(\theta)} \, \frac{\|A\|}{\eta}$$

Finally, we recognize $\kappa(A) = \sigma_1 \cdot \dfrac{1}{\sigma_n} = \|A\|\,\|A^\dagger\|$ (in this case), and we have

$$\kappa(\vec{b} \mapsto \vec{x}) = \frac{\kappa(A)}{\eta\cos(\theta)}. \quad \square$$

That concludes the "easy" parts of the proof...

Part#3: Perturbations in A — A Help-Result

Perturbations in A affect the least squares problem in two ways

- The mapping of $\mathbb{C}^n$ onto $\mathrm{range}(A)$ is distorted.
- $\mathrm{range}(A)$ is also altered.

The changes in $\mathrm{range}(A)$ introduce a "tilt" of the space; and the question is *what is the maximal tilt $\delta\alpha$ induced by a perturbation δA?*

The image of the unit sphere in $\mathbb{R}^n$, $\mathbb{S}^{n-1}$ is $A\mathbb{S}^{n-1}$, a hyper-ellipse that "lies flat" in $\mathrm{range}(A)$. $(\mathbb{S}^{n-1} = \{\vec{x} \in \mathbb{R}^n : \|\vec{x}\| = 1\})$

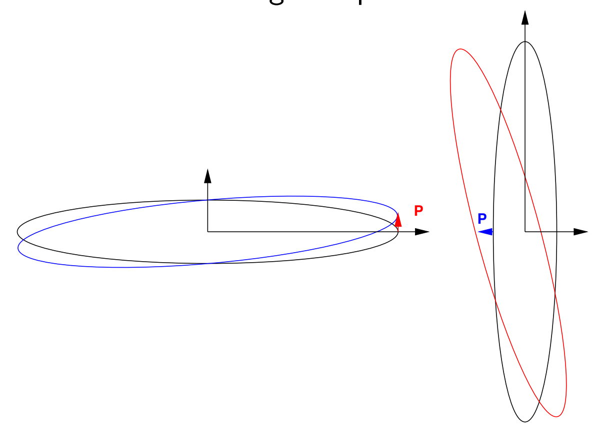
We grab a point $\vec{p} = A\vec{v}$ on the hyper-ellipse (hence $\|\vec{v}\| = 1$, since $\vec{v} \in \mathbb{S}^{n-1}$); we introduce a perturbation $\delta\vec{p} \perp \mathrm{range}(A)$.

We can express this as a rank-1 matrix perturbation $\delta A = (\delta\vec{p})\vec{v}^* \Leftrightarrow (\delta A)\vec{v} = \delta\vec{p}$, and $\|\delta A\| = \|\delta\vec{p}\|$.

SAN DIEGO STATE UNIVERSITY

Part#3: Perturbations in A — A Help-Result

Now, clearly, if we want to maximize the tilt, we should grab the hyper-ellipse as close to the origin as possible



Hence, we let $\vec{p} = \sigma_n \vec{u}_n$ (the minor semi-axis in $A\mathbb{S}^{n-1}$.)

SAN DIEGO STATE UNIVERSITY

Part#3: Perturbations in A — A Help-Result

Now, since we have A in a convenient diagonal (Σ) form, $\vec{p}$ is the last column of A, $\vec{v}^* = (0, 0, \ldots, 0, 1)$, and δA is a perturbation below the diagonal in this (last) column.

$$\vec{p} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \sigma_n \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \delta\vec{p} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \hline \delta p_{n+1} \\ \vdots \\ \delta p_m \end{bmatrix}, \quad \delta A = \left[\begin{array}{cccc} 0 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \\ \hline 0 & 0 & \ldots & \delta A_{n+1,n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \delta A_{m,n} \end{array}\right]$$

the tilting angle induced by this perturbation is

$$\tan(\delta\alpha) = \frac{\|\delta\vec{p}\|}{\sigma_n}.$$

SAN DIEGO STATE UNIVERSITY

Part#3: Perturbations in A — A Help-Result

We have

$$\tan(\delta\alpha) = \frac{\|\delta\vec{p}\|}{\sigma_n}.$$

Further,

$$\|\delta\vec{p}\| = \|\delta A\|, \quad \delta\alpha \leq \tan(\delta\alpha),$$

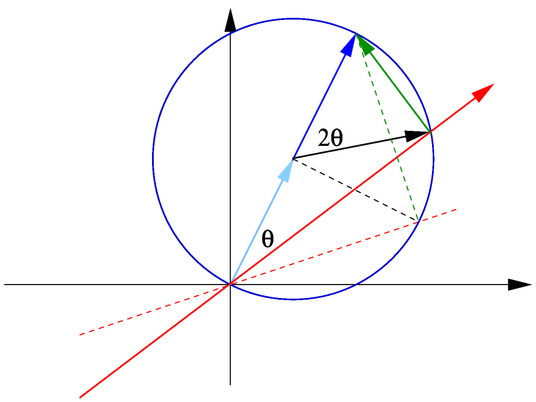
Hence,

$$\delta\alpha \leq \frac{\|\delta A\|}{\sigma_n} = \frac{\|\delta A\|}{\|A\|}\kappa(A). \qquad \boxed{\delta\alpha \leq \frac{\|\delta A\|}{\|A\|}\kappa(A)}$$

We are now ready to proceed with the proof...

SAN DIEGO STATE UNIVERSITY

Part#3: Sensitivity of $\vec{y}$ wrt. Perturbations in A — 1 of 2



Since $\vec{y}$ is the orthogonal projection of $\vec{b}$ onto $\mathrm{range}(A)$, it is determined by $\vec{b}$ and $\mathrm{range}(A)$ alone. Therefore we can study changes on $\vec{y}$ induced by tiltings $\delta\alpha$ of $\mathrm{range}(A)$.

No matter how we tilt $\mathrm{range}(A)$, $\vec{y} \in \mathrm{range}(A)$ must be orthogonal to $(\vec{b} - \vec{y}) \in \mathrm{range}(A)^{\perp}$. — As $\mathrm{range}(A)$ varies, the point $\vec{y}$ moves along a sphere of radius $\|\vec{b}\|/2$ centered at the point $\vec{b}/2$.

Part#3: Sensitivity of $\vec{y}$ wrt. Perturbations in A — 2 of 2

Tilting $\mathrm{range}(A)$ in the plane $\vec{0}$-$\vec{b}$-$\vec{y}$ by an angle $\delta\alpha$ changes the angle "2θ" at the central point $\vec{b}/2$ by $2\delta\alpha$.

The corresponding change $\delta\vec{y}$, is the base of an isosceles triangle with central angle $2\delta\alpha$, and edge length $\|\vec{b}\|/2$. Hence, $\|\delta\vec{y}\| = \|\vec{b}\|\, \sin(\delta\alpha)$

Tilting $\mathrm{range}(A)$ in any other plane results in a similar geometry in a different plane and perturbations smaller by a factor as small as $\sin\theta$.

For arbitrary perturbations we have

$$\|\delta\vec{y}\| \le \|\vec{b}\|\, \sin(\delta\alpha) \le \|\vec{b}\|\, \delta\alpha$$

Combining with previous results give us $\kappa(A \mapsto \vec{y})$

$$\|\delta\vec{y}\| \le \|\vec{b}\| \frac{\|\delta A\|}{\|A\|}\kappa(A) = \frac{\|\vec{y}\|}{\cos(\theta)} \frac{\|\delta A\|}{\|A\|}\kappa(A) \quad \Leftrightarrow \quad \frac{\|\delta\vec{y}\|}{\|\vec{y}\|} \bigg/ \frac{\|\delta A\|}{\|A\|} \le \frac{\kappa(A)}{\cos(\theta)}. \quad \square$$

Part#4: Sensitivity of $\vec{x}$ wrt. Perturbations in A — 1 of 5

We now analyze the most interesting relationship of the theorem; the sensitivity of the least squares solution to perturbations in A.

We write perturbations in two parts

$$\delta A = \begin{bmatrix} \delta A_1 \\ \hdashline \delta A_2 \end{bmatrix}, \quad \delta A_1 \in \mathbb{C}^{n\times n},\ \delta A_2 \in \mathbb{C}^{(m-n)\times n}$$

First, we look at the effects of δA_1: these perturbations change the mapping of A in its range, **but does not change $\mathrm{range}(A)$ itself**, and hence not $\vec{y}$. We get

$$(A_1 + \delta A_1)\vec{x} = \vec{b}_1$$

The condition number for this operation is simply (as before)

$$\kappa(A_1 \mapsto \vec{x}) = \frac{\|\delta\vec{x}\|}{\|\vec{x}\|} \bigg/ \frac{\|\delta A_1\|}{\|A_1\|} \le \kappa(A_1) = \kappa(A)$$

Part#4: Sensitivity of $\vec{x}$ wrt. Perturbations in A — 2 of 5

Next, we consider the effects of δA_2. This perturbation tilts $\mathrm{range}(A)$ without changing the mapping of A within this space.

The point vector $\vec{b}_1$, and the point $\vec{y} = [\vec{b}_1^* \ \vec{0}^*]^*$ are perturbed, but A_1 is not. This corresponds to perturbing $\vec{b}_1$ in $\vec{x} = A_1^{-1}\vec{b}_1$, for which the condition number takes the form

$$\kappa = \frac{\|\delta\vec{x}\|}{\|\vec{x}\|} \bigg/ \frac{\|\delta\vec{b}_1\|}{\|\vec{b}_1\|} \le \frac{\kappa(A_1)}{\eta(A_1, \vec{x})} = \frac{\kappa(A)}{\eta}$$

since...

$$\frac{\|\delta\vec{x}\|}{\|\vec{x}\|} \bigg/ \frac{\|\delta\vec{b}_1\|}{\|\vec{b}_1\|} \le \frac{\|J(\vec{x})\|}{\|\vec{x}\| \,/\, \|\vec{b}_1\|} = \frac{\|A_1^{-1}\|\, \|\vec{b}_1\|}{\|\vec{x}\|} = \frac{1}{\sigma_n} \frac{\|A_1\vec{x}\|}{\|\vec{x}\|} = \frac{\sigma_1}{\sigma_n} \frac{\|A_1\vec{x}\|}{\|A_1\|\, \|\vec{x}\|}$$

Part#4: Sensitivity of $\vec{x}$ wrt. Perturbations in A

In order to close this argument out, we must relate $\delta \vec{b}_1$ to δA_2...

The vector $\vec{b}_1$ is $\vec{y}$ expressed in the coordinates of $\mathrm{range}(A)$. Therefore, the only changes in $\vec{y}$ that are realized as changes in $\vec{b}_1$ are those that are parallel to $\mathrm{range}(A)$.

Part#4: Sensitivity of $\vec{x}$ wrt. Perturbations in A

- If $\mathrm{range}(A)$ is tilted by $\delta\alpha$ in the $\vec{0}$-$\vec{b}$-$\vec{y}$ plane, the resulting perturbation $\delta\vec{y}$ is not parallel to $\mathrm{range}(A)$, but at an angle $\left(\frac{\pi}{2} - \theta\right)$, therefore
$$\|\delta \vec{b}_1\| = \|\delta\vec{y}\| \sin\theta \leq \|\vec{b}\|\, \delta\alpha \sin\theta.$$
- If $\mathrm{range}(A)$ is tilted in a direction orthogonal to the $\vec{0}$-$\vec{b}$-$\vec{y}$ plane, $\delta\vec{y}$ is parallel to $\mathrm{range}(A)$, and we get $\|\delta\vec{y}\| \leq \|\vec{b}\|\, \delta\alpha \sin\theta$, and since $\|\delta\vec{b}_1\| \leq \|\delta\vec{y}\|$, we have
$$\|\delta \vec{b}_1\| \leq \|\vec{b}\|\, \delta\alpha \sin\theta, \quad \text{same argument as for } \kappa(A \mapsto \vec{y}).$$

We now have all the pieces to the puzzle... all we need is a bit of glue!

Part#4: Sensitivity of $\vec{x}$ wrt. Perturbations in A

Since $\|\vec{b}_1\| = \|\vec{b}\| \cos(\theta)$ we can rewrite the previous inequality as
$$\frac{\|\delta\vec{b}_1\|}{\|\vec{b}_1\|} \leq \delta\alpha \tan(\theta).$$
using the final result on slide 20 in the form
$$\frac{\delta\alpha\, \|A\|}{\|\delta A\|} \leq \kappa(A)$$
we have
$$\frac{\|\delta\vec{x}\|}{\|\vec{x}\|} \Big/ \frac{\|\delta A_2\|}{\|A\|} = \frac{\|\delta\vec{b}_1\|}{\|\vec{b}_1\|} \frac{\kappa(A)}{\eta} \frac{\|A\|}{\|\delta A_2\|} \leq \frac{\tan(\theta)\,\kappa(A)}{\eta} \frac{\delta\alpha\, \|A\|}{\|\delta A\|} \leq \frac{\tan(\theta)\,\kappa(A)^2}{\eta}$$
Adding this to the contribution from δA_1 gives us
$$\kappa(A \mapsto \vec{x}) = \kappa(A) + \frac{\tan(\theta)\,\kappa(A)^2}{\eta}. \quad \square$$

One Final Comment

Clearly, finding the least squares solution $\vec{x}$ is a tough problem:

— The condition number contains the square of the condition number of the matrix A:
$$\kappa(A \mapsto \vec{x}) = \kappa(A) + \frac{\tan(\theta)\,\kappa(A)^2}{\eta}.$$

— Even for moderately ill-conditioned matrices, the least squares problem quickly becomes very ill-conditioned.

Next time we connect the conditioning results derived here with the stability (or lack thereof) of some numerical algorithms applied to the least squares problem.

Homework #6 — Due Date in Canvas/Gradescope

TB-18.1: *see Trefethen-&-Bau for problem statement*

PB-14.1: Consider the vector $\vec{x} \in \mathbb{R}^{101}$ consisting of equi-spaced points in the interval $[0, 1]$, e.g. **x = linspace(0,1,101)';** and let $A_k \in \mathbb{R}^{101\times(k+1)}$ be the matrix consisting of columns formed by component-wize powers $\{0, \dots, k\}$ of the x-values (a Vandermonde Matrix). Let $c_\ell = \kappa(A_\ell)$ be components of the vector $\vec{c}$ containing the collection of condition numbers for these matrices. Let $\ell \in \{0, \dots, L\}$, and **make L large enough that you see something interesting.**

- Plot $\vec{c}$ (use a log scale)
- We could use these matrices (A_k) to least-squares-fit polynomials (of matching degree k) to some data-set with 101 measurements. Is it necessarily better to have more model parameters (*i.e.* fitting a higher degree polynomial)? — Discuss.

Warning: Definitions/Implementations may vary —
⇝ https://en.wikipedia.org/wiki/Vandermonde_matrix
⇝ https://www.mathworks.com/help/matlab/ref/vander.html
⇝ https://numpy.org/doc/stable/reference/generated/numpy.vander.html

Homework AI-Policy Spring 2024

AI-era Policies — SPRING 2024

AI-3 Documented: *Students can use AI in any manner for this assessment or deliverable, but they must provide appropriate documentation for all AI use.*

This applies to ALL MATH-543 WORK during the SPRING 2024 semester.

The goal is to leverage existing tools and resources to generate HIGH QUALITY SOLUTIONS to all assessments.

You MUST document what tools you use and HOW they were used (including prompts); AND how results were VALIDATED.

BE PREPARED to DISCUSS homework solutions and AI-strategies. **Participation in the in-class discussions will be an essential component of the grade for each assessment.**