Student Learning Targets, and Objectives

**Target**  The Singular Value Decomposition  
  **Objective**  Existence and Uniqueness statements  
  **Objective**  Impact: "diagonalizability"

**Target**  The SVD ↔ Matrix Properties  
  **Objective**  rank, range, null-space, norms  
  **Objective**  relation to eigenvalues, determinant  
  **Objective**  Linearly Optimal Low Rank Approximations

Recap

- Hölder and Cauchy-[Bunyakovsky]-Schwarz inequalities:
  \[ |\vec{v}^* \vec{w}| \leq \|\vec{v}\|_p \|\vec{w}\|_q, \quad \frac{1}{p} + \frac{1}{q} = 1, \quad |\vec{v}^* \vec{w}| \leq \|\vec{v}\|_2 \|\vec{w}\|_2 \]

- Bounds on the norms of matrix products
  \[ \|AB\| \leq \|A\| \|B\| \]

- General matrix norms: The Frobenius norm \( \|A\|_F^2 = \sum_{ij} |a_{ij}|^2 \)
- A geometrical introduction to the SVD.
- The reduced vs. the full SVD.
5. The Singular Value Decomposition — (6/26)

If it follows that the image of the unit sphere under we can show that every matrix \(A\) has a SVD, then it follows that the image of the unit sphere under any linear map is a hyper-ellipse...

We now turn our attention to showing that this indeed is the case...

**Figure:** The unit-sphere \(S^2\), and the image \(A S^2\), where \(A = \begin{bmatrix} 1.3127 & 0.6831 & 0.6124 \\ 0.0129 & 1.0928 & 0.6085 \\ 0.3840 & 0.0353 & 1.0158 \end{bmatrix}\).

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5. The Singular Value Decomposition — (5/26)

5. The Singular Value Decomposition — (8/26)
Theorem: $A = USV^*$

We have $\|S\|_2 \geq \sqrt{\sigma_1^2 + \tilde{w}^* \tilde{w}}$, and $S = U_1^* AV_1$. Since $U_1$ and $V_1$ are unitary, we must have $\|S\|_2 = \|A\|_2 = \sigma_1$.

Therefore $\|\tilde{w}\|^2 = \tilde{w}^* \tilde{w} = 0$, which means $\tilde{w} = 0$, hence

$$U_1^* AV_1 = S = \begin{bmatrix} \sigma_1 & 0^* \\ 0 & B \end{bmatrix}, \quad \iff A = U_1 \begin{bmatrix} \sigma_1 & 0^* \\ 0 & B \end{bmatrix} V_1^*$$

If $m = 1$, or $n = 1$, we are done. Otherwise, the sub-matrix $B$ describes the action of $A$ on the subspace orthogonal to $\tilde{v}_1$.

We can now recursively (inductively) apply the same process to $B$, and establish existence of the SVD of $A$:

$$A = U_1 \begin{bmatrix} 1 & 0^* \\ 0 & U_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0^* \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} 1 & 0^* \\ 0 & V_2 \end{bmatrix} V_1^* = U \Sigma V^*.$$

Since $\|A\|_2 = \sigma_1$, $\|A \tilde{v}_2\|_2 \leq \sigma_1$; but this must be an equality, otherwise since for some $\theta$

$$\tilde{w}_1 = \cos(\theta) \tilde{v}_1 + \sin(\theta) \tilde{v}_2, \quad \tilde{v}_1 \perp \tilde{v}_2, \quad \cos^2(\theta) + \sin^2(\theta) = 1$$

we would have $\|A \tilde{w}_1\|_2 < \sigma_1$.

This vector $\tilde{v}_2$ is a second right singular vector corresponding to the singular value $\sigma_1$; it will lead to the appearance of a $\tilde{y}$ (the last $(n-1)$ elements of $V_1^* \tilde{v}_2$) with $\|\tilde{y}\|_2 = 1$, and $\|B \tilde{y}\|_2 = \sigma_1$.

Hence, if the singular vector $\tilde{v}_1$ is not unique, then the corresponding singular value $\sigma_1$ is not simple ($\sigma_1 \neq \sigma_2$). Therefore there cannot exist a vector $\tilde{w}_1$ as above.

Now, the uniqueness of the remaining singular vectors follows by induction. □

**Recap**

**Existence and Uniqueness of the SVD**

The SVD: $A = USV^*$

**Proof**

**Bold Statement**

SVD enables us to say that every matrix is “diagonal” — as long as we use the proper bases for the domain $\in \mathbb{C}^n$, and range (image) $\in \mathbb{C}^m$ spaces.

**Changing Bases — Rotating the Map!**

Any $\tilde{b} \in \mathbb{C}^m$ can be expanded in the basis of the left singular vectors of $A$ (i.e. the columns of $U$), and any $\tilde{x} \in \mathbb{C}^n$ in the basis of the right singular vectors of $A$ (i.e. the columns of $V$)...

The coordinates for these expansions are

$$\tilde{b} = U^* \tilde{b}, \quad \tilde{x} = V^* \tilde{x}.$$

Now, the relation $\tilde{b} = A \tilde{x}$ can be written in terms of $\tilde{b'}$ and $\tilde{x'}$:

$$\tilde{b} = A \tilde{x} \iff U^* \tilde{b} = U^* A \tilde{x} = U^* \Sigma V^* \tilde{x} \iff \tilde{b'} = \Sigma \tilde{x'}.$$
The idea of diagonalizing a matrix by a change of basis is the foundation for the study of eigenvalues.

A non-defective square matrix $A$ can be expressed as a diagonal matrix of eigenvalues $\Lambda$, if the range (image) and domain are expressed in a basis of the eigenvectors. The eigenvalue decomposition of $A \in \mathbb{C}^{m \times m}$ is

$$A = X \Lambda X^{-1}$$

where $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_m)$, and the columns of $X \in \mathbb{C}^{m \times m}$ contain linearly independent eigenvectors of $A$. We can change basis for the expression $\vec{b} = A \vec{x}$:

$$\vec{b}' = X^{-1} \vec{b}, \quad \vec{x}' = X^{-1} \vec{x}.$$ 

and find that

$$\vec{b}' = \Lambda \vec{x}'$$

The SVD matrix properties

The SVD has many connections with other fundamental topics in linear algebra...

In the following slides, assume that $A \in \mathbb{C}^{m \times n}$, let $p = \min(m, n)$, and let $r \leq p$ denote the number of non-zero singular values of $A$; finally let $\text{span}(\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m)$ denote the space spanned by the vectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_m$, i.e. all linear combinations of the vectors.

Theorem (Rank of a Matrix)

$$\text{rank}(A) = r.$$ 

Proof (Rank of a Matrix)

The rank of a diagonal matrix is the number of non-zero entries. In the decomposition $A = U \Sigma V^*$, both $U$ and $V$ are full rank. Therefore $\text{rank}(A) = \text{rank}(\Sigma) = r$. □

The SVD and Eigenvalue Decomposition

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(Typical) Application Relevance

Behavior of $A$ itself, or $A^{-1}$. Behavior of $A^k$, $e^{tA}$.

Theorem (Range (Image) and Nullspace of a Matrix)

$$\text{range}(A) = \text{span}(\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r),$$ 

$$\text{null}(A) = \text{span}(\vec{v}_{r+1}, \vec{v}_{r+2}, \ldots, \vec{v}_n).$$

Proof (Range (Image) and Nullspace of a Matrix)

This follows directly from the change of bases induced by $A = U \Sigma V^*$ and the fact that

$$\text{range}(\Sigma) = \text{span}(\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_r) \subseteq \mathbb{C}^m,$$

$$\text{null}(\Sigma) = \text{span}(\vec{e}_{r+1}, \vec{e}_{r+2}, \ldots, \vec{e}_n) \subseteq \mathbb{C}^n.$$
Theorem (Euclidean and Frobenius Matrix Norms)

\[ \|A\|_2 = \sigma_1, \quad \text{and} \quad \|A\|_F = \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_r^2}. \]

Proof (Euclidean and Frobenius Matrix Norms)

We already established that \( \sigma_1 = \|A\|_2 \) in the existence proof, since \( A = U\Sigma V^* \) with unitary \( U \) and \( V \),

\[ \|A\|_2 = \|\Sigma\|_2 = \max\{\sigma_i\} = \sigma_1. \]

Now, since the Frobenius norm is invariant under unitary transformations, \( \|A\|_F = \|\Sigma\|_F = \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_r^2}. \)

The non-zero singular values of \( A \) are the square roots of the non-zero eigenvalues of \( A^*A \) or \( AA^* \) (these two matrices have the same non-zero eigenvalues).

Proof ()

From

\[ A^*A = (U\Sigma V^*)(U\Sigma V^*) = V\Sigma^*U^*U\Sigma V^* = V(\Sigma^*\Sigma)V^* = V(\Sigma^*\Sigma)V^{-1} \]

we see that \( A^*A \) and \( \Sigma^*\Sigma = \text{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_p^2) \) have the same eigenvalues, \( \lambda_i = \sigma_i^2, \ i = 1, 2, \ldots, p. \)

If \( n > p \), we have an additional \((n - p)\) zero eigenvalues.

The same argument works for \( AA^* \) (just substitute \( m \) for \( n \)).

Theorem

If \( A = A^* \), then the singular values of \( A \) are the absolute values of the eigenvalues of \( A \).

Proof (part 1)

The eigenvalues of a Hermitian matrix are real since if \((\lambda, \vec{v})\) is an eigenvalue-eigenvector pair \((\lambda \neq 0)\), then

\[ \langle \vec{v}, A\vec{v} \rangle = \langle A^*\vec{v}, \vec{v} \rangle = \langle A\vec{v}, \vec{v} \rangle \]

Hence, \( \lambda = \lambda^* \Rightarrow \lambda \in \mathbb{R} \). Further, a Hermitian matrix has a complete set of orthogonal eigenvectors. This means that we can diagonalize \( A \)

\[ A = Q\Lambda Q^* = Q(\Lambda|\text{sign}(\Lambda)|)Q^* \]

for some unitary matrix \( Q \) and \( \Lambda \) a real diagonal matrix...

Proof (part 2)

Since \( \text{sign}(\Lambda)Q^* \) is unitary, we have

\[ A = \begin{pmatrix} \rho_1 & 0 & \cdots & 0 \\ 0 & \rho_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \rho_p \end{pmatrix} \begin{pmatrix} |\rho_1| & \text{sign}(\rho_1) & \cdots & 0 \\ 0 & |\rho_2| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & |\rho_p| \end{pmatrix} \]

a SVD of \( A \), where \( \sigma_i = |\lambda_i|, i = 1, 2, \ldots, p. \) (An appropriate ordering of the columns of \( U \) guarantees that the singular values are ordered in decreasing order.) \( \square \)
The SVD ⊑ Matrix Properties

**Theorem**

For $A \in \mathbb{C}^{m \times m}$, $|\det(A)| = \prod_{i=1}^{m} \sigma_i$.

**Proof (Magnitude of Determinant is Product of Singular Values)**

$$|\det(A)| = |\det(U\Sigma V^*)| = |\det(U)||\det(\Sigma)||\det(V^*)| = 1 \cdot |\det(\Sigma)| \cdot 1 = |\det(\Sigma)| = \prod_{i=1}^{m} \sigma_i$$

where we have used the fact that $\det(AB) = \det(A)\det(B)$ and that the magnitude of the determinant of a unitary matrix is one.

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The SVD ⊑ Matrix Properties

**Theorem (Optimal Low-Rank Approximation)**

For any $\nu$ with $0 \leq \nu < r$, define

$$A_\nu = \sum_{k=1}^{\nu} \sigma_k \bar{u}_k \bar{v}_k^*$$

if $\nu = p = \min(m, n)$, define $\sigma_{\nu+1} = 0$. Then

$$\|A - A_\nu\|_2 = \inf_{B \in \mathbb{C}^{m \times n}} \|A - B\|_2 = \sigma_{\nu+1}$$

Given the SVD of $A$, $A = U\Sigma V^*$, we can represent $A$ as a sum of $r$ rank-one matrices

$$A = \sum_{k=1}^{r} \sigma_k \bar{u}_k \bar{v}_k^*$$

This is certainly not the only way to write $A$ as a sum of rank-one matrices: it could be written as a sum of its $m$ rows, $n$ columns, or even its $mn$ entries...

The decomposition above has the special property that if we truncate the sum at $\nu < r$, then that partial sum captures as much “energy” of $A$ as possible for a rank-$\nu$ sub-matrix of $A$.

We formalize this in a theorem...

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The SVD ⊑ Matrix Properties

**Proof (Optimal Low-Rank Approximation)**

Suppose that there is some $B$ with $\text{rank}(B) \leq \nu$ such that

$$\|A - B\|_2 < \|A - A_\nu\|_2 = \sigma_{\nu+1}.$$ 

Then there is an $(n - \nu)$-dimensional subspace $\text{null}(B) = W \subseteq \mathbb{C}^n$ such that $\bar{w} \in W \Rightarrow B\bar{w} = 0$. Thus $\forall \bar{w} \in W$:

$$\|A\bar{w}\|_2 = \|(A - B)\bar{w}\|_2 \leq \|A - B\|_2 \|\bar{w}\|_2 < \sigma_{\nu+1} \|\bar{w}\|_2.$$ 

Now, $W$ is an $(n - \nu)$-dimensional subspace where $\|A\bar{w}\|_2 < \sigma_{\nu+1} \|\bar{w}\|_2$. But there is a $(\nu + 1)$-dimensional subspace where $\|A\bar{w}\|_2 \geq \sigma_{\nu+1} \|\bar{w}\|_2$ — $\forall \bar{v} = \text{span}(u_1, \ldots, u_{\nu+1})$ the space spanned by the first $(\nu + 1)$ right singular vectors of $A$.

Since the sum of the dimensions of the two subspaces $(\nu + 1) + (n - \nu) = (n + 1)$ exceeds $n$, there must be a non-zero vector lying in both. This is a contradiction.
The preceding theorem has a nice geometrical interpretation.

Ponder the issue of finding the best approximation of an $n$-dimensional hyper-ellipsoid.

⇒ The best approximation by a 2-dimensional ellipse must be the ellipse spanned by the largest and second largest axis.

⇒ We get the best 3-dimensional approximation by adding the span of the 3rd largest axis, etc...

This is useful in many applications, e.g. signal compression (images, audio, etc.), analysis of large data sets, etc.

We state the following theorem, and leave the proof as an "exercise."

**Theorem**

*For the matrix $A_\nu$ as defined in the previous theorem*

$$
\| A - A_\nu \|_F = \inf_{B \in \mathbb{C}^{m \times n} \text{ rank}(B) \leq \nu} \| A - B \|_F = \sqrt{\sigma^2_{\nu+1} + \sigma^2_{\nu+2} + \cdots + \sigma^2_r}
$$

We will get back to how to compute the SVD later. For now, we note that it is a powerful tool which can be used to

- find the numerical rank of a matrix;
- find the orthonormal basis for the range (image) and null-space;
- computing $\| A \|_2$;
- computing low-rank approximations.

The SVD shows up in least squares fitting, regularization, intersection of subspaces (video games?), and many, many other problems.