What We Swept Under The Rug...

So far we have not discussed stability in a systematic way...
... unless vigorous hand-waving and “proof by picture” qualifies as systematic.

We now turn our attention to the issue of **stability**, and look at several things:
- Conditioning — sensitivity to perturbations.
- Finite precision floating point arithmetic — representation errors.
- Stability of Algorithms.

**Conditioning** \(\sim\) perturbation behavior of the mathematical problem.

**Stability** \(\sim\) perturbation behavior of an algorithm.

Outline

1. **Looking Back...**
   - Things We Ignored, or Waved Our Hands At
     - Conditioning
   - Conditioning of a Problem — “The Intrinsic Difficulty”

2. **Conditioning Examples**
   - Cancellation; Polynomial Roots; Matrix Eigenvalues
   - Conditioning of Fundamental Linear Algebra Operations

The Condition of a Problem

For us a (linear algebra) problem is a function

\[
f : X \to Y
\]

where \(X \subseteq \mathbb{C}^n\), and \(Y \subseteq \mathbb{C}^m\). Given some input \(\bar{x} \in X\), we produce an answer \(\bar{y} \in Y\).

A **well-conditioned problem** has the property that small perturbations (changes) in \(\bar{x}\) leads to small changes in \(\bar{y} = f(\bar{x})\).

An **ill-conditioned problem** has the property that small perturbations (changes) in \(\bar{x}\) leads to large changes in \(\bar{y} = f(\bar{x})\).

Clearly, we must quantify what “small” and “large” mean...
Absolute Condition Number

Let $\delta \bar{x}$ denote a small perturbation of $\bar{x}$, and let

$$\delta f \overset{\text{def}}{=} f(\bar{x} + \delta \bar{x}) - f(\bar{x})$$

be the corresponding change in $f$.

The **absolute condition number** $\hat{\kappa}(\bar{x})$ of the problem $f$ at $\bar{x}$ is defined as

$$\hat{\kappa}(\bar{x}) = \lim_{\Delta \rightarrow 0} \sup_{\|\delta \bar{x}\| \leq \Delta} \frac{\|\delta f\|}{\|\delta \bar{x}\|}$$

**Think:** the supremum over small perturbations.

For **notational convenience** we usually drop the limit, and write

$$\hat{\kappa}(\bar{x}) = \sup_{\delta \bar{x}} \frac{\|\delta f\|}{\|\delta \bar{x}\|}$$

with the understanding that $\delta \bar{x}$ and $\delta f$ are infinitesimal.

Relative Condition Number

The **relative condition number** $\kappa(\bar{x})$ of the problem is defined as

$$\kappa(\bar{x}) = \lim_{\Delta \rightarrow 0} \sup_{\|\delta \bar{x}\| \leq \Delta} \frac{\|\delta f\|}{\|\delta \bar{x}\|} \frac{1}{\|f(\bar{x})\|}$$

or, compactly,

$$\kappa(\bar{x}) = \sup_{\delta \bar{x}} \frac{\|\delta f\|}{\|f(\bar{x})\|} \frac{1}{\|\bar{x}\|}$$

If/When $f$ is differentiable we get

$$\kappa(\bar{x}) = \frac{\|\bar{x}\|}{\|f(\bar{x})\|} \|J(\bar{x})\|.$$

Differentiability and the Absolute Condition Number

If $f$ is differentiable, we can define the **Jacobian**

$$J(\bar{x}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial fn}{\partial x_1} & \frac{\partial fn}{\partial x_2} & \cdots & \frac{\partial fn}{\partial x_n} \end{bmatrix}.$$ 

Now, we have

$$\delta f = J(\bar{x}) \delta \bar{x}, \quad \|\delta \bar{x}\| \rightarrow 0,$$

and we can express the condition number in terms of the Jacobian

$$\hat{\kappa} = \|J(\bar{x})\|.$$

Absolute vs. Relative Condition Numbers

The **relative condition number** tends to be the more useful description of error propagation in numerical analysis.

Part of the reason is that errors introduced due to floating point arithmetic during computations are “relative.”

Another reason is that even if the absolute condition number is small, the relative condition number can still be large, if $\|f(\bar{x})\|$ is small. Here, a small absolute perturbation of $f(\bar{x})$ may make the result $f(\bar{x} + \delta \bar{x})$ almost completely independent of $f(\bar{x})$, i.e. completely dominated by the perturbation.

**Rules of Thumb:** If $\kappa \sim 1, 10, 10^2$ then it is “small,” and the problem is well-conditioned; if $\kappa \sim 10^6, 10^{16}$ then it is “large,” and the problem is ill-conditioned.
Example: Quantifying “Cancellation Error” 1 of 2

We consider the problem \( f : \mathbb{C}^2 \rightarrow \mathbb{C} \) defined by

\[
   f(\bar{x}) = x_1 - x_2, \quad \bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{C}^2
\]

The Jacobian of \( f \) is

\[
   J(\bar{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1 & -1 \end{bmatrix}.
\]

If we use the \( \infty \)-norm, we have \( \|J(\bar{x})\|_\infty = 2 \), and

\[
   \kappa(\bar{x}) = \frac{\|J(\bar{x})\|_\infty}{\|f(\bar{x})\|_\infty} = \frac{2 \max\{|x_1|, |x_2|\}}{|x_1 - x_2|}.
\]

Now, if \( x_1 \approx x_2 \), the problem is clearly ill-conditioned; otherwise it is well-conditioned.

Wilkinson’s Classic Example 1 of 4

Finding the roots of a polynomial given the polynomial coefficients is a classic example of an ill-conditioned problem, e.g.

\[
   x^2 - 2x + 1 = (x - 1)^2 \\
   x^2 - 2x + 0.9999 = (x - 0.99)(x - 1.01) \\
   x^2 - 2x + 0.999999 = (x - 0.999)(x - 1.001)
\]

Here, a perturbation of \( 10^{-4} \) (\( 10^{-6} \)) in one coefficient moved both roots \( 10^{-2} \) (\( 10^{-3} \)).

The roots can change \( \propto \sqrt{\delta\text{(coeff)}} \), and since

\[
   \lim_{\delta(\text{coeff}) \to 0} \frac{d}{d[\delta\text{(coeff)}]} \sqrt{\delta\text{(coeff)}} = \lim_{\delta(\text{coeff}) \to 0} \frac{1}{2\sqrt{\delta\text{(coeff)}}} \to \infty
\]

the condition number \( \kappa = \infty \).

Looking Back...
Conditioning Examples
Cancellation; Polynomial Roots; Matrix Eigenvalues
Conditioning of Fundamental Linear Algebra Operations

Example: Quantifying “Cancellation Error” 2 of 2

Wilkinson’s Classic Example 2 of 4

Even when the roots are single (distinct), polynomial root-finding is ill-conditioned:

If the \( i \)th coefficient \( a_i \) of a polynomial \( p(x) \) is perturbed by \( \delta a_i \), the perturbation of the \( j \)th root \( x_j \) is

\[
   \delta x_j = -\frac{(\delta a_i)x_j^i}{p'(x_j)}, \quad \text{and} \quad \kappa_{ji} = \frac{|a_jx_j^{i-1}|}{|p'(x_j)|}
\]

where \( \kappa_{ji} \) is the condition number of \( x_j \) with respect to perturbation of the coefficient \( a_i \).

This number can be very large.
Wilkinson’s Classic Example

The classic example is the roots of Wilkinson’s polynomial
\[ p(x) = \prod_{i=1}^{20} (x - i) = a_0 + a_1 x + \cdots + a_{19} x^{19} + x^{20} \]
with the unperturbed roots \{1, 2, \ldots, 19, 20\}.
It turns out that the most sensitive root is \( r_{15} = 15 \), and it is most sensitive to perturbations in \( a_{15} \approx -1.67 \times 10^9 \), with \( \kappa_{15,15} \approx 4.6602 \times 10^{12} \).

The figure on the next slide shows the distribution of roots of 100 randomly perturbed Wilkinson polynomials. The coefficients have been perturbed \( \tilde{a}_i = (1 + 10^{-10} r_i) a_i \), where \( r_i \) is drawn from the \( N(0,1) \) distribution (mean zero, variance 1).

Polynomial Roots: Comments

It turns out that polynomial root finding does not have to be as ill-conditioned as we have described. The ill-conditioning as described is largely associated with the unfortunate basis of expansion \( \{ x^k \}_{k=0,1,\ldots} \)
Using, e.g., the Chebyshev polynomial basis \( \{ T_n(x) \}_{n=0,1,\ldots} \) can improve the conditioning significantly.


Eigenvalues of a Non-Symmetric Matrix

Consider the two matrices
\[ A_1 = \begin{bmatrix} 1 & 1000 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 10^{-3} \\ 10^{-3} & 1 \end{bmatrix} \]
The eigenvalues and eigenvectors of \( A_1 \) are
\[ \lambda = \{1, 1\}, \quad \tilde{u}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \tilde{u}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \]
The eigenvalues and eigenvectors of \( A_2 \) are
\[ \lambda = \{2, 0\}, \quad \tilde{u}_1 = \begin{bmatrix} 0.999999500000037 \\ 0.0000009999950000 \end{bmatrix}, \quad \tilde{u}_2 = \begin{bmatrix} 0.9999999500000037 \\ 0.00009999950000 \end{bmatrix} \]
Clearly, this problem is quite ill-conditioned. If \( A \) is symmetric, then the eigenvalues are well-conditioned, with \( \kappa = \|A\|_2 / \|\lambda\| \).
Next, we take a closer look at the conditioning of the basic building blocks of linear algebra

- Matrix-Vector multiplications, \( y := Ax \)
- A Matrix, \( A \in \mathbb{C}^{m \times n} \)
- A system of equations, \( Ax = b \)

From the definition of the condition number we have

\[
\delta \text{ perturbations}
\]

Computing building blocks of linear algebra

Let \( A \in \mathbb{C}^{m \times n} \), \( x \in \mathbb{C}^{n} \), and consider the product \( Ax \). Now, consider perturbations \( \delta x \) only.

From the definition of the condition number we have

\[
\kappa(x) = \sup_{\|\delta x\|} \frac{\|A(x + \delta x) - Ax\|}{\|\delta x\|} = \sup_{\|\delta x\|} \frac{\|A\delta x\|}{\|x\|}
\]

that is

\[
\kappa(x) = \frac{\|A\|}{\|Ax\|}
\]

If \( A \) is square and non-singular, we can use the fact\(^1\) that \( \frac{\|x\|}{\|Ax\|} \leq \|A^{-1}\| \)
and get a bound independent of \( x \)

\[
\kappa(x) \leq \|A\| \|A^{-1}\|
\]

\(^1\) \( \|x\| = \|A^{-1}Ax\| \leq \|A^{-1}\| \|Ax\| \).

**Note:** If \( A \in \mathbb{C}^{m \times n} \) is a full-rank \((m \geq n)\) non-square matrix, then the previously stated results hold with \( A^{-1} \) replaced by the pseudo-inverse, e.g.

\[
A^\dagger = (A^*A)^{-1}A^*
\]

I.e.

\[
\kappa \leq \|A\| \|A^\dagger\|
\]

With this particular pseudo-inverse we have

\[
\kappa \leq \|A\|_2 \|(A^*A)^{-1}A^*\|_2 \leq \|A\|_2 \|(A^*A)^{-1}\|_2 \|A^*\|_2
\]

\[
= \frac{1}{\sigma_n} \sigma_1^2 = \left( \frac{\sigma_1}{\sigma_n} \right)^2
\]
The Condition Number of a Matrix

The product $\|A\| \|A^{-1}\|$ is ubiquitous in numerical analysis, and has its own name — the condition number of the matrix $A$;

$$\kappa(A) = \|A\| \|A^{-1}\|, \quad \text{relative to the norm } \| \cdot \|.$$ 

In this instance, the condition number is attached to the matrix $A$, not (as earlier) to a problem.

If $\kappa(A)$ is small the matrix is well-conditioned, otherwise ill-conditioned.

If $\| \cdot \| = \| \cdot \|_2$, then

$$\|A\| = \sigma_1, \quad \|A^{-1}\| = 1/\sigma_m, \quad \text{thus } \kappa(A) = \frac{\sigma_1}{\sigma_m}.$$ 

When $A$ is singular, $\kappa(A) = \infty$.

### Condition of a System of Equations

We have considered $A\vec{x} = \vec{b}$ where $A$ was fixed, and we perturbed either $\vec{x}$ or $\vec{b}$ and looked at the effect on $\vec{b}$ or $\vec{x}$.

Now, let’s perturb $A \rightarrow A + \delta A$, while holding $\vec{b}$ fixed, and study the effect on $\vec{x}$, we must have

$$(A + \delta A)(\vec{x} + \delta \vec{x}) = \vec{b}$$

$\delta A\vec{x} + A\delta \vec{x} + \delta A\vec{x} = \vec{b}$ expanded

$\delta A\vec{x} + A\delta \vec{x} + \delta A\vec{x} = 0$ used $A\vec{x} = \vec{b}$

$\delta A\vec{x} + A\delta \vec{x} = 0$ dropped doubly infinitesimal term

Now,

$$\delta \vec{x} = -A^{-1}(\delta A\vec{x}).$$
Condition of a System of Equations 2 of 2

From $\delta \bar{x} = -A^{-1}(\delta A \bar{x})$ we get

$$
\begin{align*}
\| \delta \bar{x} \| & \leq \| A^{-1} \| \| \delta A \| \| \bar{x} \| \\
\frac{\| \delta \bar{x} \|}{\| \bar{x} \|} & \leq \| A^{-1} \| \| \delta A \| \\
\frac{\| \delta \bar{x} \|}{\| \bar{x} \|} & \frac{\| \bar{x} \|}{\| \bar{x} \|} \leq \| A^{-1} \| \| A \|
\end{align*}
$$

Hence, the condition number of the problem of computing $\bar{x} = A^{-1} \bar{b}$, with respect to perturbations in $A$, is bounded by $\kappa(A)$.

From the earlier discussion, we know that the condition number of the problem of computing $\bar{x} = A^{-1} \bar{b}$, with respect to perturbations in $\bar{b}$, is bounded by $\kappa(A)$.

Homework #5 — Due Friday March 25, 2016

Trefethen-&-Bau-12.3(a,b,c)

Note for HW#4: Please, don’t print any 80 × 80-matrices!