Numerical Matrix Analysis
Notes #14 — Conditioning and Stability:
Least Squares Problems: Conditioning

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Outline

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   - Introduction: Projection, Pseudo-Inverse
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Student Learning Targets, and Objectives

Target Derivation of the Four Condition Numbers of the Least Squares Problems

Objective ...
Objective ...

Target ...
Objective ...
Recap: Last Time

Backward Stability of Back-Substitution

We looked at a backward stability proof in gory detail. — The technique is quite straightforward, albeit somewhat tedious.

— We replace the floating point operators $\oplus$, $\ominus$, $\otimes$, and $\oslash$ with exact mathematical operations $+$ relative error terms, i.e. $(x \oplus y) \leadsto (x + y)(1 + \epsilon)$, where $|\epsilon| \leq \varepsilon_{\text{mach}}$.

— Then we interpret the error as perturbations on the appropriate part of the problem formulation (so that the computed solution is the exact solution to a nearby problem).
Recap: Last Time

As we used the backward substitution algorithm for the detailed backward stability proof; we now turn to least squares problems for a detailed discussion on conditioning...

...and we recall that Accuracy(conditioning, stability), so these are all important pieces in the larger "numerics jigsaw puzzle."

Rewind (Computational Accuracy)

Suppose a backward stable algorithm is applied to solve a problem $f : X \mapsto Y$ with condition number $\kappa$ in a floating point environment satisfying the floating point representation axiom, and the fundamental axiom of floating point arithmetic. Then the relative errors satisfy

$$\frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|} = O(\kappa(x)\varepsilon_{\text{mach}}).$$
Least Squares Problems...

Once again, we return to the least squares problem.

\[ b = Ax = Pb \]

\[ r = b - Ax \]

This is easily the most technical lecture of the semester. Grab a barrel of coffee, and enjoy the ride!

We measure everything in the two-norm, and let \( \| \cdot \| = \| \cdot \|_2 \); formally we are trying to solve

**Given** \( A \in \mathbb{C}^{m \times n} \) of full rank, \( m \geq n \), \( \vec{b} \in \mathbb{C}^m \), find \( \vec{x} \in \mathbb{C}^n \) such that \( \| \vec{b} - A\vec{x} \|_2 \) is minimized.
The conditioning of these problems depend on a combination of

1. The conditioning of square systems of equations
2. The geometry of orthogonal projections.

The topic is subtle, and has nontrivial implications for the stability (and ultimately, the accuracy) of least squares algorithms.

From our previous discussion of least squares problem we know

$$\vec{x} = A^\dagger \vec{b}, \text{ where } A^\dagger = (A^* A)^{-1} A^*, \text{ or } R^{-1} Q^*, \text{ or } V \Sigma^{-1} U^*$$

$$A \vec{x} = \vec{y}, \text{ where } \vec{y} = P \vec{b}, \text{ P } = AA^\dagger$$

$P$ is the orthogonal projector onto $\text{range}(A)$, and $A^\dagger \in \mathbb{C}^{m \times m}$ is the pseudo-inverse of $A$. For this theoretical infinite-precision discussion the choice of implementation/expression for the pseudo-inverse does not matter.
Least Squares Problems... Conditioning

Conditioning is the measure of sensitivity of solutions to perturbations in the data.

Our data are

\[ A \in \mathbb{C}^{m \times n}, \quad \text{and} \quad \vec{b} \in \mathbb{C}^{m}, \]

and the solution is either the vector \( \vec{x} \in \mathbb{C}^{n} \), or the vector \( \vec{y} = P\vec{b} \) (depending on our point of view / application).

We end up with four combinations of input/output-perturbations:

<table>
<thead>
<tr>
<th>Input, Output</th>
<th>( \vec{y} \in \mathbb{C}^{m} )</th>
<th>( \vec{x} \in \mathbb{C}^{n} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \vec{b} \in \mathbb{C}^{m} )</td>
<td>( \kappa(\vec{b} \mapsto \vec{y}) )</td>
<td>( \kappa(\vec{b} \mapsto \vec{x}) )</td>
</tr>
<tr>
<td>( A \in \mathbb{C}^{m \times n} )</td>
<td>( \kappa(A \mapsto \vec{y}) )</td>
<td>( \kappa(A \mapsto \vec{x}) )</td>
</tr>
</tbody>
</table>
Three Dimensionless Parameters

We are going to express all the condition-numbers using three dimensionless parameters — $\kappa(A)$, $\theta$, and $\eta$

$\kappa(A)$ is our old friend the condition number of the matrix $A$

$$\kappa(A) = \frac{\sigma_1}{\sigma_n}.$$  

$\theta$ is the angle between $\vec{b}$ and $\vec{y} = A\vec{x} = P\vec{b}$,
Three Dimensionless Parameters

\( \eta \) is a measure of how much \( \| \vec{y} \| \) falls short of its maximum value, given \( \| A \| \) and \( \| \vec{x} \| \): (or how misaligned (\( \vec{y}, \vec{x} \)) is with (\( \vec{u}_1, \vec{v}_1 \)). — Implications for “Model Quality”)

\[
\eta = \frac{\| A \| \| \vec{x} \|}{\| \vec{y} \|} = \frac{\| A \| \| \vec{x} \|}{\| A \vec{x} \|} = \sigma_1 \frac{\| \vec{x} \|}{\| A \vec{x} \|}.
\]

These parameters lie in the ranges

\[
\kappa(A) \in [1, \infty), \quad \theta \in \left[ 0, \frac{\pi}{2} \right], \quad \eta \in \left[ 1, \kappa(A) \right),
\]

and

\[
\cos(\theta) = \frac{\| \vec{y} \|}{\| \vec{b} \|} \in [0, 1], \quad \theta = \cos^{-1} \left( \frac{\| \vec{y} \|}{\| \vec{b} \|} \right).
\]

Usually, this is the quantity of interest; not \( \theta \) itself.

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Theorem (Conditioning of Least Squares Problems)

Let \( \mathbf{b} \in \mathbb{C}^m \) and \( A \in \mathbb{C}^{m \times n} \) of full rank be given.

The least squares problem, \( \min_{\mathbf{x} \in \mathbb{C}^n} \| \mathbf{b} - A\mathbf{x} \| \) has the following 2-norm relative condition numbers describing the sensitivities of \( \mathbf{y} \) and \( \mathbf{x} \) to perturbations in \( \mathbf{b} \) and \( A \):

\[
\begin{array}{c|c|c}
\downarrow \text{Input, Output} & \mathbf{y} \in \mathbb{C}^m & \mathbf{x} \in \mathbb{C}^n \\
\hline
\mathbf{b} \in \mathbb{C}^m & \frac{1}{\cos(\theta)} & \frac{\kappa(A)}{\eta \cos(\theta)} \\
A \in \mathbb{C}^{m \times n} & \frac{\kappa(A)}{\cos(\theta)} & \kappa(A) + \frac{\kappa(A)^2 \tan(\theta)}{\eta} \\
\end{array}
\]

The results in the first row are exact, being attained for certain perturbations \( \delta \mathbf{b} \), and the results in the second row are upper bounds.
A Note on the Theorem

In the special case $m = n$, the least squares problem reduces to a square non-singular problem, with $\theta = 0$, and the table looks like

\[
\begin{array}{c|cc}
\downarrow \text{Input, Output} & \vec{y} & \vec{x} \\
\hline
\vec{b} & 1 & \frac{\kappa(A)}{\eta} \\
A & 0 & \kappa(A) \\
\end{array}
\]

Since $A$ is square + full rank, $\vec{y}$ is already in the range, so no projection is needed; hence the condition number is 0.

**Note:** Condition numbers less than 1 are rare, and usually indicate that there is no relation between the input and the output.
(Massively) Simplifying the Proof Using the SVD

We have argued (a long, long time ago) that every matrix has a singular value decomposition.

Let $U\Sigma V^* = A$ be the SVD of $A$. We can use $U$ and $V$ to get two convenient bases in which we prove the theorem. Since 2-norm perturbations are not changed by a unitary change of basis, the perturbation behavior of $A$ is the same as that of $\Sigma$.

**Without loss of generality** we can assume that

\[
A = \begin{bmatrix}
\Sigma \\
0
\end{bmatrix} = \begin{bmatrix}
A_1 \\
0
\end{bmatrix} = \begin{bmatrix}
\sigma_1 & & & \\
& \sigma_2 & & \\
& & \ddots & \\
0 & & & \sigma_n
\end{bmatrix}
\]
Moving Along...

Now with

\[ \mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{bmatrix}, \quad \mathbf{b}_1 \in \mathbb{C}^n, \quad \mathbf{b}_2 \in \mathbb{C}^{m-n} \]

the projection of \( \mathbf{b} \) onto \( \text{range}(A) \) is trivial

\[ \mathbf{y} = P\mathbf{b} = \begin{bmatrix} \mathbf{b}_1 \\ 0 \end{bmatrix} \]

Now, \( A\mathbf{x} = \mathbf{y} \) has the unique solution

\[ \mathbf{x} = A_1^{-1}\mathbf{b}_1. \]

We note that the orthogonal projector, and the pseudo-inverse of \( A \) take the forms

\[ P = \begin{bmatrix} I_{n \times n} & 0 \\ 0 & 0 \end{bmatrix}, \quad A^\dagger = \begin{bmatrix} A_1^{-1} & 0 \end{bmatrix}. \]
Part #1: Sensitivity of $\vec{y}$ wrt. Perturbations in $\vec{b}$

$\vec{y} = P \vec{b}$ is a linear differentiable map; and the Jacobian is $P$ itself, with $\|P\| = 1$.

For a differentiable map $x \mapsto f(\vec{x})$ the condition number is

$$\kappa(\vec{x}) = \frac{\|J(\vec{x})\|}{\|f(\vec{x})\| / \|\vec{x}\|}.$$ 

Here we have

$$\kappa(\vec{b} \mapsto \vec{y}) = \frac{\|P\|}{\|\vec{y}\| / \|\vec{b}\|} = \frac{1}{\cos(\theta)}. \quad \Box$$
Part #2: Sensitivity of \( \vec{x} \) wrt. Perturbations in \( \vec{b} \)

\[ \vec{x} = A^\dagger \vec{b} \]

is also linear, with Jacobian \( J = A^\dagger \), so

\[ \kappa(\vec{b} \mapsto \vec{x}) = \frac{||A^\dagger||}{||\vec{x}||/||\vec{b}||} = ||A^\dagger|| \frac{||\vec{b}||}{||\vec{y}||} \frac{||\vec{y}||}{||\vec{x}||} = ||A^\dagger|| \frac{1}{\cos(\theta)} \frac{||A||}{\eta} \]

Finally, we recognize

\[ \kappa(A) = \sigma_1 \cdot \frac{1}{\sigma_n} = ||A|| \cdot ||A^\dagger|| \] (in this case),

and we have

\[ \kappa(\vec{b} \mapsto \vec{x}) = \frac{\kappa(A)}{\eta \cos(\theta)}. \quad \square \]

That concludes the “easy” parts of the proof...
Part #3: Perturbations in $A$

Perturbations in $A$ affect the least squares problem in two ways:

- The mapping of $\mathbb{C}^n$ onto $\text{range}(A)$ is distorted.
- $\text{range}(A)$ is also altered.

The changes in $\text{range}(A)$ introduce a “tilt” of the space; and the question is *what is the maximal tilt $\delta \alpha$ induced by a perturbation $\delta A$?*

The image of the unit sphere in $\mathbb{R}^n$, $\mathbb{S}^{n-1}$ is $A\mathbb{S}^{n-1}$, a hyper-ellipse that “lies flat” in $\text{range}(A)$. ($\mathbb{S}^{n-1} = \{ \vec{x} \in \mathbb{R}^n : \| \vec{x} \| = 1 \}$)

We grab a point $\vec{p} = A\vec{v}$ on the hyper-ellipse (hence $\| \vec{v} \| = 1$, since $\vec{v} \in \mathbb{S}^{n-1}$); we introduce a perturbation $\delta \vec{p} \perp \text{range}(A)$.

We can express this as a rank-1 matrix perturbation $\delta A = (\delta \vec{p})\vec{v}^* \Leftrightarrow (\delta A)\vec{v} = \delta \vec{p}$, and $\| \delta A \| = \| \delta \vec{p} \|$. 
Part #3: Perturbations in $A$

Now, clearly, if we want to maximize the tilt, we should grab the hyper-ellipse as close to the origin as possible.

Hence, we let $\tilde{p} = \sigma_n \tilde{u}_n$ (the minor semi-axis in $A S^{n-1}$.)
Now, since we have $A$ in a convenient diagonal ($\Sigma$) form, $\vec{p}$ is the last column of $A$, $\vec{v}^* = (0, 0, \ldots, 0, 1)$, and $\delta A$ is a perturbation below the diagonal in this (last) column.

$$\vec{p} = \begin{bmatrix}
0 \\
\vdots \\
0 \\
\sigma_n \\
\hline
0 \\
\vdots \\
0
\end{bmatrix}, \quad \delta \vec{p} = \begin{bmatrix}
0 \\
\vdots \\
0 \\
\sigma_n \\
\hline
\delta p_{n+1} \\
\vdots \\
\delta p_m
\end{bmatrix}, \quad \delta A = \begin{bmatrix}
0 & 0 \\
0 & \ddots \\
0 & 0 & \cdots & \delta A_{n+1,n} \\
\vdots & \vdots & \ddots & \ddots \\
0 & 0 & \cdots & \delta A_{m,n}
\end{bmatrix}$$

the tilting angle induced by this perturbation is

$$\tan(\delta \alpha) = \frac{\|\delta \vec{p}\|}{\sigma_n}.$$
Part#3: Perturbations in $A$

We have

$$\tan(\delta \alpha) = \frac{\|\delta \vec{p}\|}{\sigma_n}.$$ 

Further,

$$\|\delta \vec{p}\| = \|\delta A\|, \quad \delta \alpha \leq \tan(\delta \alpha),$$ 

Hence,

$$\delta \alpha \leq \frac{\|\delta A\|}{\sigma_n} = \frac{\|\delta A\|}{\|A\|} \kappa(A).$$

$$\delta \alpha \leq \frac{\|\delta A\|}{\|A\|} \kappa(A)$$

We are now ready to proceed with the proof...
Part#3: Sensitivity of $\vec{y}$ wrt. Perturbations in $A$

Since $\vec{y}$ is the orthogonal projection of $\vec{b}$ onto range$(A)$, it is determined by $\vec{b}$ and range$(A)$ alone. Therefore we can study changes on $\vec{y}$ induced by tiltings $\delta \alpha$ of range$(A)$.

No matter how we tilt range$(A)$, $\vec{y} \in \text{range}(A)$ must be orthogonal to $(\vec{b} - \vec{y}) \in \text{range}(A)^\perp$. — As range$(A)$ varies, the point $\vec{y}$ moves along a sphere of radius $\|\vec{b}\|/2$ centered at the point $\vec{b}/2$. 
Part#3: Sensitivity of $\vec{y}$ wrt. Perturbations in $A$

Tilting range$(A)$ in the plane $\vec{0} - \vec{b} - \vec{y}$ by an angle $\delta\alpha$ changes the angle “$2\theta$” at the central point $\vec{b}/2$ by $2\delta\alpha$.

The corresponding change $\delta\vec{y}$, is the base of an isosceles triangle with central angle $2\delta\alpha$, and edge length $\|\vec{b}\|/2$. Hence, $\|\delta\vec{y}\| = \|\vec{b}\| \sin(\delta\alpha)$

Tilting range$(A)$ in any other plane results in a similar geometry in a different plane and perturbations smaller by a factor as small as $\sin \theta$.

For arbitrary perturbations we have

$$\|\delta\vec{y}\| \leq \|\vec{b}\| \sin(\delta\alpha) \leq \|\vec{b}\| \delta\alpha$$

Combining with previous results give us $\kappa(A \mapsto \vec{y})$

$$\|\delta\vec{y}\| \leq \|\vec{b}\| \frac{\|\delta A\|}{\|A\|} \kappa(A) = \frac{\|\vec{y}\|}{\cos(\theta)} \frac{\|\delta A\|}{\|A\|} \kappa(A) \iff \frac{\|\delta\vec{y}\|}{\|\vec{y}\| / \|A\|} \leq \frac{\kappa(A)}{\cos(\theta)}. \quad \Box$$
Part#4: Sensitivity of $\vec{x}$ wrt. Perturbations in $A$

We now analyze the most interesting relationship of the theorem; the sensitivity of the least squares solution to perturbations in $A$.

We write perturbations in two parts

$$\delta A = \begin{bmatrix} \delta A_1 \\ \delta A_2 \end{bmatrix}, \quad \delta A_1 \in \mathbb{C}^{n \times n}, \ \delta A_2 \in \mathbb{C}^{(m-n) \times n}$$

First, we look at the effects of $\delta A_1$: these perturbations change the mapping of $A$ in its range, but does not change $\text{range}(A)$ itself, and hence not $\vec{y}$. We get

$$(A_1 + \delta A_1)\vec{x} = \vec{b}_1$$

The condition number for this operation is simply (as before)

$$\kappa(A_1 \mapsto \vec{x}) = \frac{\|\delta \vec{x}\|}{\|\vec{x}\|} \sqrt{\frac{\|\delta A_1\|}{\|A_1\|}} \leq \kappa(A_1) = \kappa(A)$$
Next, we consider the effects of $\delta A_2$. This perturbation tilts \( \text{range}(A) \) without changing the mapping of \( A \) within this space.

The point vector $\vec{b}_1$, and the point $\vec{y} = [\vec{b}_1^* \vec{0}^*]^*$ are perturbed, but $A_1$ is not. This corresponds to perturbing $\vec{b}_1$ in $\vec{x} = A_1^{-1} \vec{b}_1$, for which the condition number takes the form

\[
\kappa = \frac{\|\delta \vec{x}\|}{\|\vec{x}\|} \left/ \frac{\|\delta \vec{b}_1\|}{\|\vec{b}_1\|} \right. \leq \frac{\kappa(A_1)}{\eta(A_1, \vec{x})} = \frac{\kappa(A)}{\eta}
\]

since...

\[
\frac{\|\delta \vec{x}\|}{\|\vec{x}\|} \left/ \frac{\|\delta \vec{b}_1\|}{\|\vec{b}_1\|} \right. \leq \frac{\|J(\vec{x})\|}{\|\vec{x}\| / \|\vec{b}_1\|} = \frac{\|A_1^{-1}\| \|\vec{b}_1\|}{\|\vec{x}\|} = \frac{1}{\sigma_n} \frac{\|A_1 \vec{x}\|}{\|\vec{x}\|} = \frac{\sigma_1}{\sigma_n} \frac{\|A_1 \vec{x}\|}{\|A_1\| \|\vec{x}\|}
\]
In order to close this argument out, we must relate $\delta \vec{b}_1$ to $\delta A_2$...

The vector $\vec{b}_1$ is $\vec{y}$ expressed in the coordinates of $\text{range}(A)$. Therefore, the only changes in $\vec{y}$ that are realized as changes in $\vec{b}_1$ are those that are parallel to $\text{range}(A)$. 
Part #4: Sensitivity of $\vec{x}$ wrt. Perturbations in $A$

- If $\text{range}(A)$ is tilted by $\delta \alpha$ in the $\vec{0}$-$\vec{b}$-$\vec{y}$ plane, the resulting perturbation $\delta \vec{y}$ is not parallel to $\text{range}(A)$, but at an angle $(\frac{\pi}{2} - \theta)$, therefore

$$\|\delta \vec{b}_1\| = \|\delta \vec{y}\| \sin \theta \leq \|\vec{b}\| \delta \alpha \sin \theta.$$ 

- If $\text{range}(A)$ is tilted in a direction orthogonal to the $\vec{0}$-$\vec{b}$-$\vec{y}$ plane, $\delta \vec{y}$ is parallel to $\text{range}(A)$, and we get $\|\delta \vec{y}\| \leq \|\vec{b}\| \delta \alpha \sin \theta$, and since $\|\delta \vec{b}_1\| \leq \|\delta \vec{y}\|$, we have

$$\|\delta \vec{b}_1\| \leq \|\vec{b}\| \delta \alpha \sin \theta,$$  

same argument as for $\kappa(A \mapsto \vec{y})$.

We now have all the pieces to the puzzle... all we need is a bit of glue!
Since $\|\vec{b}_1\| = \|\vec{b}\| \cos(\theta)$ we can rewrite the previous inequality as

$$\frac{\|\delta \vec{b}_1\|}{\|\vec{b}_1\|} \leq \delta \alpha \tan(\theta).$$

using the final result on slide 19 in the form

$$\frac{\delta \alpha \|A\|}{\|\delta A\|} \leq \kappa(A)$$

we have

$$\frac{\|\delta \vec{x}\|}{\|\vec{x}\|} / \|A\| = \frac{\|\delta \vec{b}_1\|}{\|\vec{b}_1\|} \frac{\kappa(A)}{\eta} \frac{\|A\|}{\|\delta A_2\|} \leq \frac{\tan(\theta) \kappa(A)}{\eta} \frac{\delta \alpha \|A\|}{\|\delta A\|} \leq \frac{\tan(\theta) \kappa(A)^2}{\eta}$$

Adding this to the contribution from $\delta A_1$ gives us

$$\kappa(A \mapsto \vec{x}) = \kappa(A) + \frac{\tan(\theta) \kappa(A)^2}{\eta}.$$
One Final Comment

Clearly, finding the least squares solution $\vec{x}$ is a tough problem:

— The condition number contains the square of the condition number of the matrix $A$.

— Even for moderately ill-conditioned matrices, the least squares problem quickly becomes very ill-conditioned.

Next time we connect the conditioning results derived here with the stability (or lack thereof) of some numerical algorithms applied to the least squares problem.
Homework #6

TB-18.1: see Trefethen-&-Bau for problem statement

PB-14.1: Consider the vector \( \vec{x} \in \mathbb{R}^{101} \) consisting of equi-spaced points in the interval \([0, 1]\), e.g. \( \vec{x} = \text{linspace}(0,1,101)' \); and let \( A_k \in \mathbb{R}^{101 \times (k+1)} \) be the matrix consisting of columns formed by component-wise powers \( \{0, \ldots, k\} \) of the \( x \)-values (a Vandermonde Matrix). Let \( c_\ell = \kappa(A_\ell) \) be components of the vector \( \vec{c} \) containing the collection of condition numbers for these matrices. Let \( \ell \in \{0, \ldots, L\} \), and make \( L \) large enough that you see something interesting.

- Plot \( \vec{c} \) (use a log scale)
- We could use these matrices \( (A_k) \) to least-squares-fit polynomials (of matching degree \( k \)) to some data-set with 101 measurements. Is it necessarily better to have more model parameters (i.e. fitting a higher degree polynomial)? — Discuss.

https://en.wikipedia.org/wiki/Vandermonde_matrix