Numerical Matrix Analysis
Lecture Notes #24 — The Singular Value Decomposition
Application to Signal and Data Analysis

Peter Blomgren,
⟨blomgren.peter@gmail.com⟩
Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720
http://terminus.sdsu.edu/
Spring 2016

Outline

1. The Singular Value Decomposition
   - Google Hits
   - Some Applications

2. Dynamic Pattern Formation

Searches on scholar.google.com


**Applications/Keywords**

**“Random” results from scholar.google.com**

**Canonical.Correlation.Analysis**

fMRI, Neural activity, Climate forecasts, Identification of hydrological neighborhoods, El Niño/Southern Oscillation (ENSO) prediction, ...

**Proper.Orthogonal.Decomposition**

Turbulent flows, Vibroimpact oscillations, Cavity flows, Optimal control of fluids, Magneto-Hydro-Dynamics (MHD) flows, ...

**Karhunen.Loeve**

Characterization of human faces, Cosmology, Turbulence Modeling, Multi-spectral image restoration, Universal image compression, ...

---

The Kuramoto-Sivashinsky equation, here in polar coordinates

\[
 u_t = -u_{rrrr} - \frac{1}{r^4} u_{\phi\phi\phi\phi} - \frac{2}{r^2} u_{r\phi\phi} - \frac{2}{r} u_{rr} + \frac{2}{r^3} u_r \phi \\
 - \left[ 2 - \frac{1}{r^2} \right] u_r - \left[ \frac{4}{r^4} + \frac{2}{r^2} \right] u_{\phi\phi} - \left[ \frac{1}{r^3} + \frac{2}{r} \right] u_r \\
 + \eta_1 u - \eta_2 \left[ u_r^2 + \frac{1}{r^2} u_\phi^2 \right] - \eta_3 u^3,
\]

is a model for the behavior of cellular flames stabilized on a circular porous plug burner. For different simulation parameters \((\eta_1, \eta_2, \eta_3, R)\) it exhibits a wide array of complex flame patterns; — mimicking patterns observed in physical experiments.
We defer all discussion on how to time-integrate the Kuramoto-Sivashinsky equation until Math 693b.

We note that each time step (from $t$ to $t + \delta t$, where $\delta t$ is "small"), requires the solution of several non-Hermitian linear systems $A\tilde{x} = \tilde{b}$, where in our set-up $A \in \mathbb{R}^{m \times m}$, with $m = 2048$.

In what follows, we keep the parameters $(\eta_1, \eta_2, \eta_3) = (0.32, 1.00, 0.017)$ constant, and vary only the radius of the circular burner.

For the majority of radii, we get static (non-moving) patterns, which are quite easy to classify.

However, for some fairly narrow parameter ranges we get time-dependent (dynamic) patterns. We will use the SVD to analyze and classify these patterns.

**Dynamic Pattern #1: 3-Cell (Nearly) Rigid Rotation**

**Dynamic Pattern #2: 3-Cell “Hopping Pattern”**

**Figure**: Some of the static patterns observed using the Kuramoto-Sivashinsky integration scheme. 2-cell pattern, $R = 5.0$; 3-cell pattern, $R = 6.0$; 4-cell pattern, $R = 8.0$; 6/1-cell pattern, $R = 10.0$; 8/2-cell pattern, $R = 12.0$; 10/5/1-cell pattern, $R = 14.5$; Common simulation parameters: $(\eta_1, \eta_2, \eta_3) = (0.32, 1.00, 0.017)$. 

---

*Peter Blomgren, ⟨blomgren.peter@gmail.com⟩*
Analyzing the Dynamic Patterns: The Method of Snapshots

We use the SVD in order to analyze and classify these dynamic patterns.

Each “frame”, \( u^{(i)}(r, \phi) \) with 32 radial, and 64 azimuthal points, of the sequence defines a \( 2048 \times 1 \)-vector \( \tilde{f}_i \):

\[
\tilde{f}_i = \begin{bmatrix}
    u^{(i)}(r_1, \phi_1) \\
    \vdots \\
    u^{(i)}(r_1, \phi_{64}) \\
    u^{(i)}(r_2, \phi_1) \\
    \vdots \\
    u^{(i)}(r_{32}, \phi_{64})
\end{bmatrix}
\]

Analyzing the Dynamic Patterns: The Snapshot Matrices

For both the rigidly rotating, and the hopping pattern, we have computed 7200 frames, hence for each simulation we can build a \( 2048 \times 7200 \) matrix of snapshots

\[
\tilde{A} = \begin{bmatrix}
    \tilde{f}_1 & \tilde{f}_2 & \ldots & \tilde{f}_{7200}
\end{bmatrix}.
\]

It turns out that for this application (and many others) it is advantageous to view each snapshot as a perturbation from the mean, so with \( \bar{\mathbf{m}}_f = \text{mean}_{i=1, \ldots, 7200}(\tilde{f}_i) \), we define new vectors

\[
\tilde{f}_i = \tilde{f}_i - \bar{\mathbf{m}}_f,
\]

and a new “snapshot perturbation matrix”

\[
A = \begin{bmatrix}
    \tilde{f}_1 & \tilde{f}_2 & \ldots & \tilde{f}_{7200}
\end{bmatrix}.
\]

Restatement of the obvious: \( U \) is orthonormal, and has the same column space as \( A \), i.e. it is an orthonormal basis for \( \text{range}(A) \).

The singular values \( \sigma_i \) tell us how “important” each column in \( U \) is, i.e. how much perturbation “energy” is controlled by the \( i \)th column of \( U \).
For the **rigid rotation** we see that \( \sim 70\% \) of the energy is controlled by \( \bar{u}_1 - \bar{u}_2 \), which express rotations of 3-cell perturbations from the mean. There is \( \sim 15\% \) of the energy in the \( \bar{u}_3 - \bar{u}_4 \) pair (rotations of 1-cell perturbations), and \( \sim 5\% \) of energy in 6-cell, and 2-cell perturbations; the first 10 columns catch in excess of 98\% of the energy, and hence provide an almost complete description of the motion.

---

For the **hopping motion** we first notice that the rotations of 3-cell perturbations from the mean now only control \( \sim 42\% \) of the motion, and about 5 times as much energy (\( \sim 10\% \)) has “leaked” outside the first 10 columns. — All of this is an indication that the motion is much more complex. Further, the \( \bar{u}_3 - \bar{u}_4 \) pair (of the rigid rotation) has formed a more complex \( \bar{u}_3 - \bar{u}_4 - \bar{u}_5 \) triple, and the 2-, 4-, and 5-cell rotations have overtaken the importance of the 6-cell rotations (which is no longer in the “top 10.”)

**Observation:** Since, in both cases, the first 10 basis vectors control at least 90\% of the motion

\[
\bar{f}_i \approx \sum_{k=1}^{10} a_{ik} \bar{u}_k, \quad \text{where} \quad a_{ik} = \bar{u}_k^* \bar{f}_i
\]

should be a good approximation. \( \sim \) **Compression:** By storing 10 (2048 \( \times \) 1) basis vectors, 7200 \( \times \) 10 coefficients, and the average vector (2048 \( \times \) 1) \( \bar{m}_f \) for a total of 94,528 values instead of the full 7200 \( \times \) 2048 = 14,745,600-value dataset, we get a compression ratio of \( \frac{1}{156} \).
The Coefficients $a_{ik}$

The coefficients $a_{ik} = \bar{u}_k^* \bar{f}_i$ give us a lot of useful information.

If the rotation is completely rigid then when $\bar{u}_k - \bar{u}_{k+1}$ describe the rotation of some $n$-cell pattern, the points $(a_{i,k}, a_{i,k+1})$ should form a circle in $\mathbb{R}^2$, usually referred to as phase space.

![Figure: The phase plots for $\bar{u}_1-\bar{u}_2$, $\bar{u}_3-\bar{u}_4$, and $\bar{u}_7-\bar{u}_8$ corresponding to the nearly rigid rotation. We notice a very small deformation for the $\bar{u}_3-\bar{u}_4$ phase portrait, and see that the $\bar{u}_7-\bar{u}_8$ phase portrait (controlling $\sim 4.6\%$ energy) is quite egg-shaped and has period 2.](image)

Analyzing Other Types of Data

Clearly, the SVD does not care what kind of data we encode in the matrix $A$, we can think of many applications...

- $\bar{f}_i = $ Passport photographs (face recognition)
- $\bar{f}_i = $ Finger-prints
- $\bar{f}_i = $ DNA-(sub)sequence
- $\bar{f}_i = $ Multiple simultaneous temperature readings
- $\bar{f}_i = $ Demographic data

For time-dependent data, we can look that the phase-portraits; for other types of data, the $k$-tuple of coefficients $(a_{i_1}, \ldots, a_{i_k})$ defines a “signature” of $\bar{f}_i$ expressed in the orthogonal basis. The signature may be useful for identification purposes.

Is there an SVD hiding in biometric passports?

Figure: The phase plots for the hopping state looks very different. The three pairs: $\bar{u}_1-\bar{u}_2$, $\bar{u}_3-\bar{u}_4$, and $\bar{u}_7-\bar{u}_8$ all display quasi-periodic behavior.

The comparison of the phase-diagrams for the nearly rigid rotation and the hopping state is the most straight-forward way of classifying (and distinguishing) these dynamic patterns.

Looking at all the phase-diagrams for motions in the range $R \in [7.3600, 7.7475]$ may give us an insight into how the hopping state is “born.”