Numerical Matrix Analysis
Notes #3 — Orthogonal Vectors, Matrices and Norms

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Outline

Student Learning Targets, and Objectives

Target Vectors

Objective Euclidean Inner Product: the Dot Product — Bilinearity
Objective Orthogonality and Orthonormality
Objective Linear Indepenence, Basis
Objective Projections
Objective Vector Norms

Target Matrices

Objective Symmetric and Hermitian Matrices
Objective Inverses and Hermitian Transposes for Matrix Products
Objective Unitary Matrices
Objective Matrix Norms

Target Actions

Objective Projections

Previously...

A quick review / crash course in basic linear algebra:

• Vectors: Transpose, Addition & Subtraction

• Matrix-Vector Product

• Vandermonde Matrix ... and Linear Least Squares Problems

• Matrix-Matrix Product

• Transpose of a Matrix (A^T)

• Range and Nullspace of a Matrix A

• Rank of a Matrix $A_{m \times n}$

• Inverse of a Matrix A

Introduction
Fundamental Concepts
Next...

Transpose (Adjoint) / Hermitian
Inner Products, Matrix Properties, Orthogonality
Unitary Matrices, Vector Norms, Matrix Norms

Now...   ...More Fundamental Concepts

The **Transpose** (Adjoint) a.k.a **Hermitian** (Transpose, or Conjugate) of a matrix $A \in \mathbb{C}^{m \times n}$...

For a scalar $z \in \mathbb{C}$, $z = a + bi$, the **complex conjugate** $\overline{z}$, or z^* is obtained by negating the imaginary part, *i.e.* $z^* = a - bi$.

Note that if $z \in \mathbb{R}$, then $z^* = z$.

For a matrix $A \in \mathbb{C}^{m \times n}$, the Hermitian Conjugate $A^* \in \mathbb{C}^{n \times m}$ is the matrix

$$
A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} \quad \Rightarrow \quad \mathbf{A}^* = \begin{bmatrix} a_{11}^* & a_{21}^* & a_{31}^* & a_{41}^* \\ a_{12}^* & a_{22}^* & a_{32}^* & a_{42}^* \end{bmatrix}
$$

Introduction
Fundamental Concepts
Next...

Transpose (Adjoint) / Hermitian
Inner Products, Matrix Properties, Orthogonality
Unitary Matrices, Vector Norms, Matrix Norms

The Hermitian Conjugate

If $A = A^*$, the matrix A is said to be **Hermitian**.

Note that a **Hermitian matrix must be square.**

In the case that A is real-valued, *i.e.* $A \in \mathbb{R}^{m \times n}$, then
$A = A^* = A^T$ (the Hermitian conjugate equals the **transpose**).

If $A = A^T$, the matrix A is said to be **Symmetric**.

Our book (TREFETHEN-BAU) tends to state results and theorems
in terms of complex vectors and matrices, and hence use the
Hermitian conjugate, *i.e.* $\vec{x}^*$ is a row-vector.

**The advantage of this approach is that we are able to state
the most general results.**

Note: There are some differences in regards to properties over $\mathbb{R}^n$ and $\mathbb{C}^n$; those gory details are explored in
[MATH 524]

Introduction
Fundamental Concepts
Next...

Transpose (Adjoint) / Hermitian
Inner Products, Matrix Properties, Orthogonality
Unitary Matrices, Vector Norms, Matrix Norms

The Inner Product of Two Vectors   a.k.a the **dot product**

The Euclidean **inner product**, denoted $\langle \vec{x}, \vec{y} \rangle$, of two column vectors $\vec{x}, \vec{y} \in \mathbb{C}^m$ is defined

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^* \vec{y} = \sum_{i=1}^{m} x_i^* y_i$$

note that the inner product is a scalar quantity.

The **Euclidean length**, $\|\vec{x}\|$, of $\vec{x} \in \mathbb{C}^m$ is defined
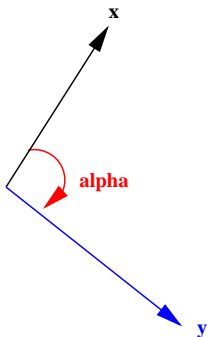
$$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle} = \sqrt{\vec{x}^* \vec{x}} = \sqrt{\sum_{i=1}^{m} |x_i|^2}$$

Introduction
Fundamental Concepts
Next...

Transpose (Adjoint) / Hermitian
Inner Products, Matrix Properties, Orthogonality
Unitary Matrices, Vector Norms, Matrix Norms

Inner Product: Geometrical Interpretation

The inner product can also be written

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^* \vec{y} = \|\vec{x}\| \cdot \|\vec{y}\| \cdot \cos(\alpha)$$

where α is the angle between $\vec{x}$ and $\vec{y}$

Introduction
Fundamental Concepts
Next...

Transpose (Adjoint) / Hermitian
Inner Products, Matrix Properties, Orthogonality
Unitary Matrices, Vector Norms, Matrix Norms

Inner Product: Properties   Bi-Linearity

The inner product is **bilinear**, *i.e.* it is linear in each argument
separately:

$$
\begin{align}
(1) \qquad (\vec{x}_1 + \vec{x}_2)^* \, \vec{y} &= \vec{x}_1^* \, \vec{y} + \vec{x}_2^* \, \vec{y} \\
(2) \qquad \vec{x}^* \, (\vec{y}_1 + \vec{y}_2) &= \vec{x}^* \, \vec{y}_1 + \vec{x}^* \, \vec{y}_2 \\
(3) \qquad (\alpha\vec{x})^*(\beta\vec{y}) &= \alpha^* \beta \, \vec{x}^* \, \vec{y}
\end{align}
$$

where $\vec{x}$, $\vec{x}_1$, $\vec{x}_2$, $\vec{y}$, $\vec{y}_1$, $\vec{y}_2 \in \mathbb{C}^m$, and α, $\beta \in \mathbb{C}$.

Compare: Bilinearity of the matrix-vector product. The Euclidean inner product is really
"just" a particular application/interpretation of the matrix-vector product.

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Introduction
Fundamental Concepts
Next...

Transpose (Adjoint) / Hermitian
Inner Products, Matrix Properties, Orthogonality
Unitary Matrices, Vector Norms, Matrix Norms

Associated Matrix Properties

For any two matrices A and B, of compatible dimensions, *i.e.*
$A \in \mathbb{C}^{m \times n}$, and $B \in \mathbb{C}^{n \times k}$ the following holds

$$(AB)^* = B^* A^*$$

If the matrices A and B are square, and invertible, the following
holds

$$(AB)^{-1} = B^{-1} A^{-1}$$

When necessary, we use the notation A^{-*} for $(A^*)^{-1} \equiv (A^{-1})^*$.

Question: What is $(AB)^{-*}$ (when well-defined)?

Introduction
Fundamental Concepts
Next...

Transpose (Adjoint) / Hermitian
Inner Products, Matrix Properties, Orthogonality
Unitary Matrices, Vector Norms, Matrix Norms

Orthogonal and Orthonormal Vectors

Two vectors are **orthogonal** if and only if $\langle \vec{x}, \vec{y} \rangle = \vec{x}^* \vec{y} = 0$,

$$0 = \frac{\vec{x}^* \vec{y}}{\|\vec{x}\| \cdot \|\vec{y}\|} = \cos(\alpha) \Leftrightarrow \alpha = \pi/2 + k \cdot \pi, \ k \in \mathbb{Z}.$$

A **set** of **non-zero** vectors S is **orthogonal** if its elements are pairwise orthogonal, *i.e.*

$$\forall \vec{x}, \vec{y} \in S, \quad \vec{x} \neq \vec{y} \quad \Rightarrow \quad \vec{x}^* \vec{y} = 0$$

A **set** of vectors S is **orthonormal** if it is **orthogonal**, and $\forall \vec{x} \in S$, $\|\vec{x}\| = 1$, *i.e.* all vectors are **unit-vectors**.

Introduction
Fundamental Concepts
Next...

Transpose (Adjoint) / Hermitian
Inner Products, Matrix Properties, Orthogonality
Unitary Matrices, Vector Norms, Matrix Norms

Linear Independence of Orthogonal Set

Theorem (Linear Independence)

The vectors in an orthogonal set S are linearly independent.

Proof (Linear Independence of Orthogonal Vectors).

If the vectors in S are not independent, then $\exists \vec{v}_k \in S : \vec{v}_k \neq \vec{0}$, so that

$$\vec{v}_k = \sum_{i \neq k} c_i \vec{v}_i.$$

Since $\vec{v}_k \neq 0$, $\langle \vec{v}_k, \vec{v}_k \rangle > 0$, now we use the bi-linearity property of inner products, and the orthogonality of S:

$$0 < \langle \vec{v}_k, \vec{v}_k \rangle = \left\langle \vec{v}_k, \sum_{i \neq k} c_i \vec{v}_i \right\rangle = \sum_{i \neq k} c_i \underbrace{\langle \vec{v}_k, \vec{v}_i \rangle}_{0, \, \forall i \neq k} = 0.$$

This contradicts the assumption that the vectors are linearly dependent, hence proving the theorem. □

Introduction
Fundamental Concepts
Next...

Transpose (Adjoint) / Hermitian
Inner Products, Matrix Properties, Orthogonality
Unitary Matrices, Vector Norms, Matrix Norms

Corollary: Basis for $\mathbb{C}^m$

Corollary

If an orthogonal set $S \subseteq \mathbb{C}^m$ contains m vectors, then it is a basis for $\mathbb{C}^m$.

We can write any vector $\vec{v} \in \mathbb{C}^m$ as a unique linear combination

$$\vec{v} = \sum_{i=1}^m a_i \vec{s}_i, \quad \text{where} \quad a_i = \frac{\langle \vec{s}_i, \vec{v} \rangle}{\|\vec{s}_i\|^2}.$$

The computation of $a_i \vec{s}_i$ is a **projection** of $\vec{v}$ onto the direction $\vec{s}_i$.

We can use this in order to decompose arbitrary vectors into orthogonal components...

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Introduction
Fundamental Concepts
Next...

Transpose (Adjoint) / Hermitian
Inner Products, Matrix Properties, Orthogonality
Unitary Matrices, Vector Norms, Matrix Norms

Orthogonal Vector Components

Suppose we have an **orthonormal set** of vectors $\{\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n\}$, $\vec{q}_i \in \mathbb{C}^m$, $n \leq m$.

Now, for any vector $\vec{v} \in \mathbb{C}^m$, the vector

$$\vec{r} = \vec{v} - \sum_{i=1}^{n} \langle\, \vec{q}_i,\, \vec{v}\,\rangle \vec{q}_i$$

is orthogonal to $\{\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_n\}$:

$$\langle\, \vec{q}_k,\, \vec{r}\,\rangle = \langle\, \vec{q}_k,\, \vec{v}\,\rangle - \underbrace{\sum_{i=1}^{n} \langle\, \vec{q}_i,\, \vec{v}\,\rangle \langle\, \vec{q}_k,\, \vec{q}_i\,\rangle}_{\langle\, \vec{q}_k,\, \vec{v}\,\rangle \underbrace{\langle\, \vec{q}_k,\, \vec{q}_k\,\rangle}_{1}} = 0.$$

Introduction
Fundamental Concepts
Next…

Transpose (Adjoint) / Hermitian
Inner Products, Matrix Properties, Orthogonality
Unitary Matrices, Vector Norms, Matrix Norms

Orthogonal Vector Components   2 of 3

We see that by applying this procedure, we have decomposed the
vector $\vec{v}$ into $(n+1)$ orthogonal components:

$$\vec{v} = \vec{r} + \sum_{i=1}^{n} \langle \vec{q}_i, \vec{v} \rangle \vec{q}_i$$

If $\{\vec{q}_i\}$ is a basis for $\mathbb{C}^m$, then $n = m$ and $\vec{r} = \vec{0}$, *i.e.*

$$\vec{v} \stackrel{!}{=} \sum_{i=1}^{n} \langle \vec{q}_i, \vec{v} \rangle \vec{q}_i = \sum_{i=1}^{n} (\vec{q}_i^* \vec{v}) \vec{q}_i = \sum_{i=1}^{n} \vec{q}_i (\vec{q}_i^* \vec{v}) = \sum_{i=1}^{n} (\vec{q}_i \vec{q}_i^*) \vec{v}$$

Introduction
Fundamental Concepts
Next...

Transpose (Adjoint) / Hermitian
Inner Products, Matrix Properties, Orthogonality
Unitary Matrices, Vector Norms, Matrix Norms

Orthogonal Vector Components   3 of 3

$$\vec{v} = \sum_{i=1}^{n} \langle \vec{q}_i, \vec{v} \rangle \vec{q}_i = \sum_{i=1}^{n} (\vec{q}_i^* \vec{v}) \vec{q}_i = \sum_{i=1}^{n} \vec{q}_i (\vec{q}_i^* \vec{v}) = \sum_{i=1}^{n} (\vec{q}_i \vec{q}_i^*) \vec{v}$$

In the expression $\sum_{i=1}^{n} (\vec{q}_i^* \vec{v}) \vec{q}_i$ we view $\vec{v}$ as a linear combination of the

vectors $\vec{q}_i$, with coefficients $(\vec{q}_i^* \vec{v})$; whereas in the mathematically

equivalent expression $\sum_{i=1}^{n} (\vec{q}_i \vec{q}_i^*) \vec{v}$, we view $\vec{v}$ as a sum of **orthogonal**

projections onto the various directions $\vec{q}_i$.

We will return to the issue of projection matrices of the formed by other
products, $\vec{q}_i \vec{q}_i^*$ soon.

Introduction
Fundamental Concepts
Next...

Transpose (Adjoint) / Hermitian
Inner Products, Matrix Properties, Orthogonality
Unitary Matrices, Vector Norms, Matrix Norms

Unitary Matrices

A square matrix $Q \in \mathbb{C}^{m \times m}$ is **unitary** (in the real case "orthogonal") if

$$Q^* = Q^{-1} \quad \Leftrightarrow \quad Q^* Q = I$$

In terms of the columns, $\vec{q}_i$ of Q this looks like

$$\begin{bmatrix} \underline{\quad} & \vec{q}_1^* & \underline{\quad} \\ \underline{\quad} & \vec{q}_2^* & \underline{\quad} \\ & \vdots & \\ \underline{\quad} & \vec{q}_m^* & \underline{\quad} \end{bmatrix} \begin{bmatrix} | & | & & | \\ \vec{q}_1 & \vec{q}_2 & \cdots & \vec{q}_m \\ | & | & & | \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

We have $\vec{q}_i^* \vec{q}_j = \delta_{ij}$, the **Kronecker delta**, equal to 1 if and only if $i = j$, and 0 otherwise.

Introduction
Fundamental Concepts
Next...

Transpose (Adjoint) / Hermitian
Inner Products, Matrix Properties, Orthogonality
Unitary Matrices, Vector Norms, Matrix Norms

Multiplication by a Unitary Matrix

Since the norms of the columns of a unitary matrix are 1, multiplication by a unitary matrix preserves the Euclidean norm, and inner product in the following sense:

For a unitary Q:

$$(1) \qquad \langle Q\vec{x}, Q\vec{y} \rangle = (Q\vec{x})^*(Q\vec{y}) = \vec{x}^* \underbrace{Q^*Q}_{I} \vec{y} = \vec{x}^* \vec{y} = \langle \vec{x}, \vec{y} \rangle$$

$$(2) \qquad \|Q\vec{x}\| = \|\vec{x}\|$$

The invariance of inner products mean that angles between vectors are preserved.

In the real case, multiplication by an orthogonal matrix corresponds to a **rigid rotation** (if $\det(Q) = 1$) or a combined **rotation − reflection** (if $\det(Q) = -1$) of the vector space.

Introduction
Fundamental Concepts
Next...

Transpose (Adjoint) / Hermitian
Inner Products, Matrix Properties, Orthogonality
Unitary Matrices, Vector Norms, Matrix Norms

Vector Norms

Norms give us the essential notion of size and distance in a vector space — these are our tools for measuring the quality of approximations and convergence in our algorithms.

Definition (Norm)

A **norm** is a function $\| \cdot \| : \mathbb{C}^m \to \mathbb{R}$ that assigns a real-valued (length) to each vector. A norm must satisfy the following three conditions for all vectors $\vec{x}, \vec{y} \in \mathbb{C}^m$, and scalars $\alpha \in \mathbb{C}$,

(1) $\|\vec{x}\| \geq 0$, and $\|\vec{x}\| = 0$ only if $\vec{x} = 0$

(2) $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

(3) $\|\alpha\vec{x}\| = |\alpha| \, \|\vec{x}\|$

(2) is known as the **"triangle inequality."**

Introduction
Fundamental Concepts
Next...

Transpose (Adjoint) / Hermitian
Inner Products, Matrix Properties, Orthogonality
Unitary Matrices, Vector Norms, Matrix Norms

The p-norms

The p-norms (sometimes referred to as the ℓ_p-norms), parametrized by p are defined by

$$\|\vec{x}\|_p = \left[\sum_{i=1}^{m} |x_i|^p\right]^{1/p}$$

As an illustration, the **unit sphere** $\|\vec{x}\|_p = 1$, $\vec{x} \in \mathbb{R}^2$ is illustrated for some common (and uncommon) p-norms, on the following slides.

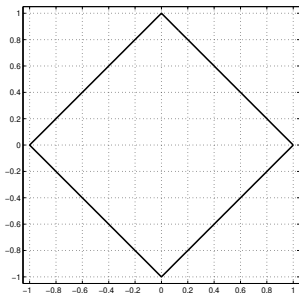2-norm the standard Euclidean length function.

1-norm sometimes referred to as the Manhattan/taxicab-distance.

0-norm counts the number of non-zero elements in a vector.

Introduction
Fundamental Concepts
Next...

Transpose (Adjoint) / Hermitian
Inner Products, Matrix Properties, Orthogonality
Unitary Matrices, Vector Norms, Matrix Norms
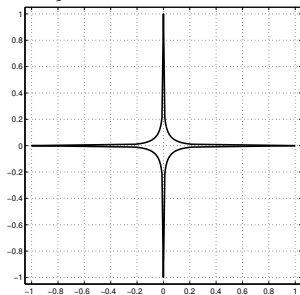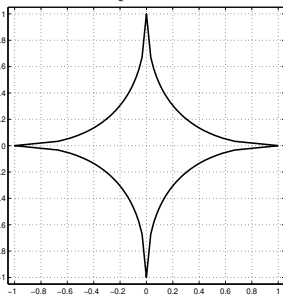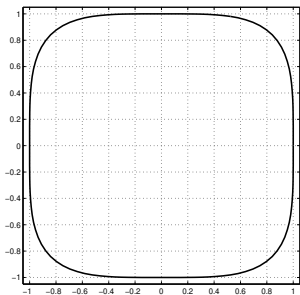
The *p*-norms

Some commonly used *p*-norms
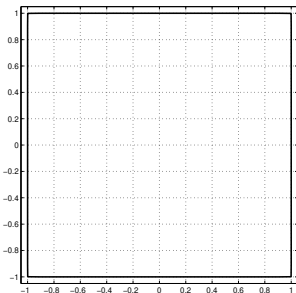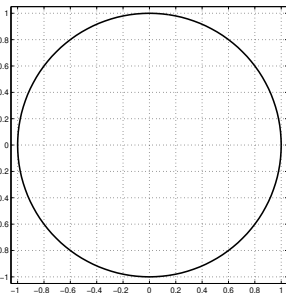
$$\|\vec{x}\|_1 = \sum_{i=1}^{m} |x_i|, \quad \|\vec{x}\|_2 = \left[\sum_{i=1}^{m} |x_i|^2\right]^{1/2}, \quad \|\vec{x}\|_\infty = \max_{i=1...m} |x_i|$$

Introduction
Fundamental Concepts
Next...

Transpose (Adjoint) / Hermitian
Inner Products, Matrix Properties, Orthogonality
Unitary Matrices, Vector Norms, Matrix Norms

The p-norms   3 of 3

Some exotic p-{norms,non-norms}

$$\|\vec{x}\|_4 = \left[\sum_{i=1}^{m} |x_i|^4\right]^{1/4}, \quad \|\vec{x}\|_{1/2} = \left[\sum_{i=1}^{m} |x_i|^{1/2}\right]^2, \quad \|\vec{x}\|_{1/4} = \left[\sum_{i=1}^{m} |x_i|^{1/4}\right]^4$$

Note: when $p < 1$ the "norms" are not convex; which means the triangle
inequality will not hold; and strictly speaking these are not norms...
$\exists$ **Movie**.

Introduction
Fundamental Concepts
Next...

Transpose (Adjoint) / Hermitian
Inner Products, Matrix Properties, Orthogonality
Unitary Matrices, Vector Norms, Matrix Norms

Weighted p-norms

The **weighted p-norms** $\| \cdot \|_{W,p}$ are derived from the p-norms:

$$\|\vec{x}\|_{W,p} = \|W\vec{x}\|_p$$

where W is *e.g.* a diagonal matrix, in which the ith diagonal entry is the weight $w_i \neq 0$:

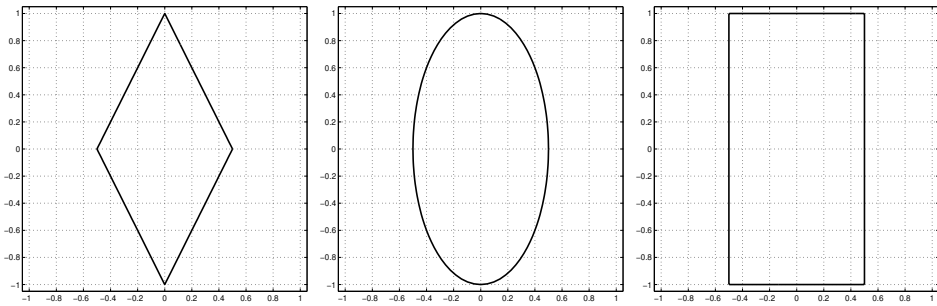$$\|\vec{x}\|_{W,p} = \left[\sum_{i=1}^{m} |w_i x_i|^p \right]^{1/p}$$

Introduction
Fundamental Concepts
Next...

Transpose (Adjoint) / Hermitian
Inner Products, Matrix Properties, Orthogonality
Unitary Matrices, Vector Norms, Matrix Norms

Weighted *p*-norms

Figure: Visualization of the unit-sphere for the weighted 1-, 2- and ∞-norms, where $W = \mathrm{diag}(2, 1)$.

The concept of weighted *p*-norms can be generalized to arbitrary non-singular weight matrices W.

Introduction
Fundamental Concepts
Next...

Transpose (Adjoint) / Hermitian
Inner Products, Matrix Properties, Orthogonality
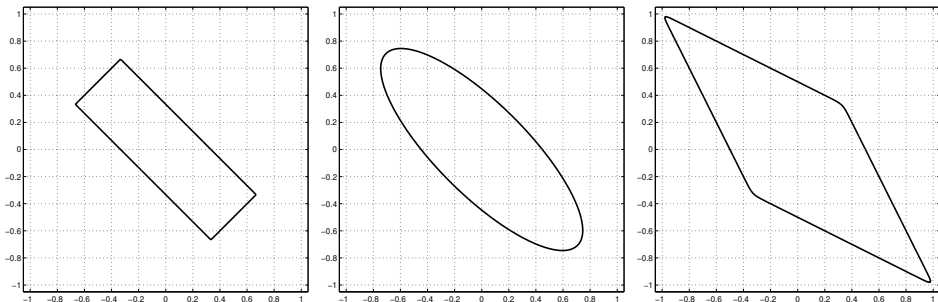Unitary Matrices, Vector Norms, Matrix Norms

Weighted *p*-norms

Figure: Visualization of the unit-sphere for the weighted 1-, 2- and ∞-norms, where $W = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

$\exists$ Movie.

Introduction
Fundamental Concepts
Next...

Transpose (Adjoint) / Hermitian
Inner Products, Matrix Properties, Orthogonality
Unitary Matrices, Vector Norms, Matrix Norms

Matrix Norms — Induced by Vector Norms

Given a vector norms $\| \cdot \|_{(m)}$ and $\| \cdot \|_{(n)}$ on the domain and range of $A \in \mathbb{C}^{m \times n}$, the induced matrix norm $\|A\|_{(m,n)}$ is

$$\|A\|_{(m,n)} = \sup_{\vec{x} \in \mathbb{C}^n - \{\vec{0}\}} \left[\frac{\|A\vec{x}\|_{(m)}}{\|\vec{x}\|_{(n)}} \right]$$
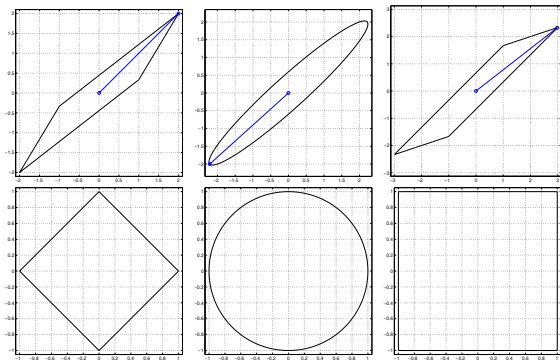
In any sane application, both $\| \cdot \|_{(m)}$ and $\| \cdot \|_{(n)}$ will be of the same type, *i.e.* the p-norms (with the same p).

Due to the linearity of norms — the third norm-condition — it is sufficient to maximize the matrix norm over $\vec{x} \in \mathbb{C}^n : \|\vec{x}\| = 1$...

Most of the time the norms with $p = 2$ are used. Indeed, if nothing else is specified, this is usually implied.

Introduction
Fundamental Concepts
Next...

Transpose (Adjoint) / Hermitian
Inner Products, Matrix Properties, Orthogonality
Unitary Matrices, Vector Norms, Matrix Norms

Illustration: Matrix Norms

$$A = \begin{bmatrix} 1 & 2 \\ 1/3 & 2 \end{bmatrix}, \quad \begin{array}{rcl} \lambda(A) & = & \{2.45743, 0.54257\} \quad \text{eigenvalues} \\ \sigma(A) & = & \{2.98523, 0.44664\} \quad \text{singular values} \end{array}$$

$$\|A\|_1 = 4 \qquad \|A\|_2 \approx 2.9852 \qquad \|A\|_\infty = 3$$

Introduction
Fundamental Concepts
Next...

Transpose (Adjoint) / Hermitian
Inner Products, Matrix Properties, Orthogonality
Unitary Matrices, Vector Norms, Matrix Norms

Special Cases: Matrix p-norms

If D is a diagonal matrix, then

$$\|D\|_p = \max_{1 \leq i \leq m} |d_i|.$$

The 1-norm of a matrix is the maximal column-abs-sum:

$$\|A\|_1 = \max_{1 \leq j \leq n} \|\vec{a}_j\|_1$$

The ∞-norm of a matrix is the maximal row-abs-sum:

$$\|A\|_\infty = \max_{1 \leq i \leq m} \|\vec{a}_i^*\|_1$$

Next Time

- Additional discussion on norms:

  - Inequalities, General matrix norms, The Frobenius norm, Bounds on norms of products of matrices.

- **The Singular Value Decomposition (SVD)**.