# Numerical Solutions to PDEs

## Lecture Notes #1 — Introduction

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## Outline



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## Academic Life — MSc

- KTH MSc. Engineering Physics, Royal Institute of Technology (KTH), Stockholm, Sweden. Thesis Advisers: Michael Benedicks, Department of Mathematics KTH, and Erik Aurell, Stockholm University, Department of Mathematics. Thesis Topic: *"A Renormalization Technique for Families with Flat Maxima."*

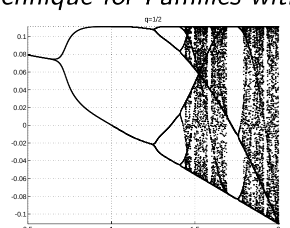


**Figure:** Bifurcation diagram for the family $f_{a,\frac{1}{2}}$ [BLOMGREN-1994]

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## Academic Life — PhD

- UCLA PhD. UCLA Department of Mathematics. Adviser: Tony F. Chan. PDE-Based Methods for Image Processing. Thesis title: *"Total Variation Methods for Restoration of Vector Valued Images."*

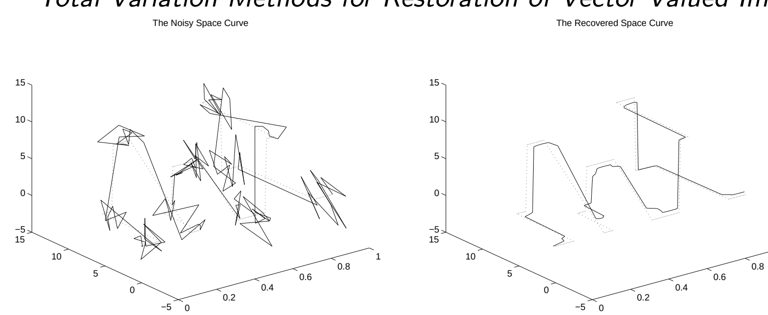


**Figure:** The noisy (SNR = 4.62 dB), and recovered space curves. Notice how the edges are recovered. [BLOMGREN-1998]

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## Academic Life — Postdoc

- Research Associate. Stanford University, Department of Mathematics. Main Focus: Time Reversal and Imaging in Random Media (with George Papanicolaou, *et. al.*)

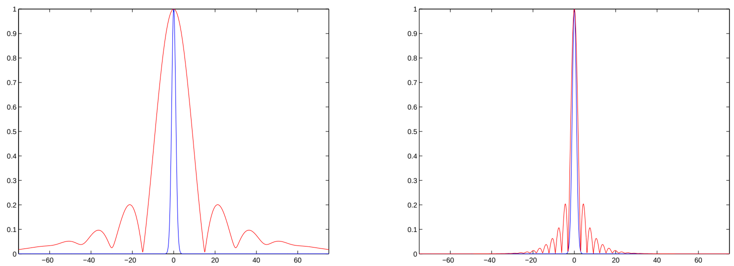


**Figure:** Comparison of the theoretical formula for a medium with $L = 600\,m$, $a_e = 195\,m$, $\gamma = 2.12 \times 10^{-5}\,m^{-1}$. [LEFT] shows a homogeneous medium, $\gamma = 0$, with $a = 40\,m$ TRM (in red / wide Fresnel zone), and a random medium with $\gamma = 2.12 \times 10^{-5}$ (in blue). [RIGHT] shows $\gamma = 0$, with $a = a_e = 195\,m$ (in red), and $\gamma = 2.12 \times 10^{-5}$, with $a = 40\,m$ (in blue). The match confirms the validity of [the theory]. [BLOMGREN-PAPANICOLAOU-ZHAO-2002]





## Academic Life — Professor

- Professor, SDSU, Department of Mathematics and Statistics. Projects: Computational Combustion, Biomedical Imaging (Mitochondrial Structures, Heartcell Contractility, Skin/Prostate Cancer Classification), carbon sequestration, compressed sensing.

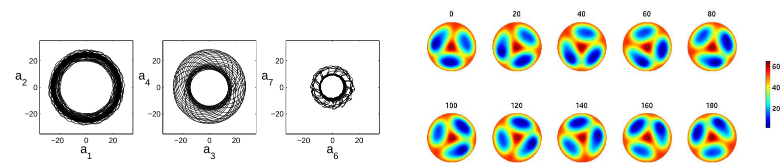


**Figure:** [LEFT] Phase-space projections produced by the time coefficients of the POD decomposition of the rotating pattern shown in [RIGHT]. [BLOMGREN-GASNER-PALACIOS-2005]





## Contact Information

| | |
|---|---|
| Office | GMCS-587 |
| Email | **blomgren.peter@gmail.com** |
| Web | **http://terminus.sdsu.edu/SDSU/Math693b/** |
| Office Hours | TuTh 11:00am – 12:00pm, W 1:00pm – 2:30pm and by appointment. |

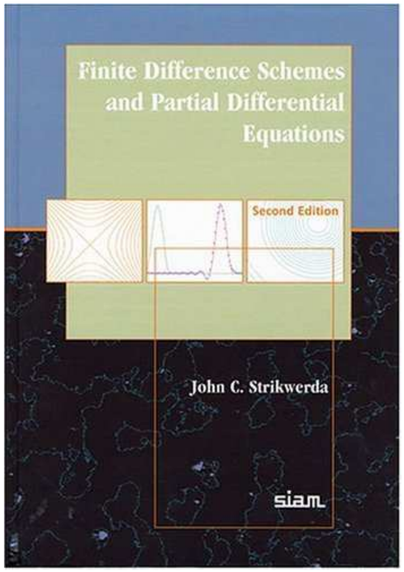




## Fun Times... ⇒ Endurance Sports

- Triathlons:
  - (13) Ironman distance (2.4 + 112 + 26.2) — 11:48:57 [PR]
  - (16) Half Ironman distance — 5:14:20
- Running
  - (1) 100k Race (62.1 miles) — 15:37:46
  - (1) Trail Double-marathon (52 miles) — 10:59:00
  - (5) Trail 50-mile races — 9:08:46
  - (8) Trail 50k (31 mile) races — 5:20:57
  - (15) Road/Trail Marathons — 3:26:19 (7:52/mi)
  - (28) Road/Trail Half Marathons — 1:36:25 (7:21/mi)

## Math 693b: Literature




**"Finite Difference Schemes and Partial Differential Equations," 2nd Edition.**

Author: John C. Strikwerda.

Publisher: Society for Industrial and Applied Mathematics.

ISBN: **0-89871-567-9**.

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**Class notes and web-page.**
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## Syllabus

| Chapter | Title |
|---|---|
| 1 | **Hyperbolic Partial Differential Equations** |
| 2 | Analysis of Finite Difference Schemes |
| 3 | Order of Accuracy of Finite Difference Schemes |
| 4 | Stability for Multistep Methods |
| 5 | Dissipation and Dispersion |
| 6 | **Parabolic Partial Differential Equations** |
| 7 | Systems of PDEs in Higher Dimensions |
| 8 | Second-Order Equations |
| 9 | Analysis of Well-Posed and Stable Problems |
| 10 | Convergence Estimates for Initial Value Problems |
| 11 | Well-Posed and Stable Initial-Boundary Value Problems |
| 12 | **Elliptic Partial Differential Equations** and Difference Schemes |
| 13 | Linear Iterative Methods |
| 14 | Steepest Descent and Conjugate Gradient Methods. |

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## Grading

| | |
|---|---|
| Homework* | 50% |
| Project$^{\times}$ | 50% |

* $\approx$ 7 assignments; first $\approx$ 3 mostly theoretical with some computational components, last $\approx$ 4 "purely" computational.

× Details to be discussed.

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## Expectations and Procedures, I

- Most class attendance is "OPTIONAL" — Homework and announcements will be posted on the class web page. If/when you attend class:
  - Please be on time.
  - Please pay attention.
  - Please turn off mobile phones.
  - Please be courteous to other students and the instructor.
  - Abide by university statutes, and all applicable local, state, and federal laws.

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## Expectations and Procedures, II

- Please, turn in assignments on time. (The instructor reserves the right not to accept late assignments.)
- The instructor will make special arrangements for students with documented learning disabilities and will `try` to make accommodations for other unforeseen circumstances, *e.g.* illness, personal/family crises, etc. in a way that is fair to all students enrolled in the class. **Please contact the instructor EARLY regarding special circumstances.**
- Students are expected **and encouraged** to ask questions in class!
- Students are expected **and encouraged** to to make use of office hours! If you cannot make it to the scheduled office hours: contact the instructor to schedule an appointment!





## Expectations and Procedures, $II\frac{1}{2}$ New Spring 2017

**Late HW Policy**

- 10% loss of value / full week late. Examples:
  - 6 days 23 hrs 59 minutes 59 seconds late = FULL VALUE
  - 7 days 0 hrs 0 min 1 sec late = 10% LOSS OF VALUE
  - 14 days 0 hrs 0 min 1 sec late = 20% LOSS OF VALUE





## Expectations and Procedures, III

- Missed midterm exams: Don't miss exams! The instructor reserves the right to schedule make-up exams, make such exams oral presentation, and/or base the grade solely on other work (including the final exam).
- Missed final exam: Don't miss the final! Contact the instructor ASAP or a grade of incomplete or F will be assigned.
- **Academic honesty**: submit your own work — but feel free to discuss homework with other students in the class!





## Honesty Pledges, I

- The following **Honesty Pledge** must be included in all programs you submit (as part of homework and/or projects):
  - I, (your name), pledge that this program is completely my own work, and that I did not take, borrow or steal code from any other person, and that I did not allow any other person to use, have, borrow or steal portions of my code. I understand that if I violate this honesty pledge, I am subject to disciplinary action pursuant to the appropriate sections of the San Diego State University Policies.
- Work missing the honesty pledge **may not be graded!**

## Honesty Pledges, II

- Larger reports must contain the following text:
  - I, (your name), pledge that this report is completely my own work, and that I did not take, borrow or steal any portions from any other person. Any and all references I used are clearly cited in the text. I understand that if I violate this honesty pledge, I am subject to disciplinary action pursuant to the appropriate sections of the San Diego State University Policies. `Your signature`.
- Work missing the honesty pledge `may not be graded!`





## Math 693b: Computer Resources

You need access to a computing environment in which to write your code; — you may want to use a combination of Matlab (for quick prototyping and short homework assignments) and C/C++ or Fortran (or something completely different).

Free C/C++ (`gcc`) and Fortran (`f77`) compilers are available for Linux/UNIX.

You may also want to consider buying the student version of Matlab: `http://www.mathworks.com/`

**SDSU students can download a copy of matlab from**
http://edoras.sdsu.edu/∼download/matlab.html
[Licensing Subject to Change With Minimal Notice]





## Math 693b: Formal Prerequisites (Graduate Bulletin)

`Math 531`, `Math 537` and `Math 693a`

531 ⇒ **PDEs**

- Boundary value problems for the heat and wave equations: eigenfunction expansions, Sturm-Liouville theory and Fourier series. D'Alembert's solution to wave equation; characteristics. Laplace's equation, maximum principles, Bessel functions.

537 ⇒ **ODEs**

- Theory of ODEs; existence and uniqueness, dependence on initial conditions and parameters, linear systems, stability and asymptotic behavior, plane autonomous systems, series solutions at regular singular points.

693a ⇒ **Advanced Numerical Analysis (Numerical Optimization)**

- Numerical optimization, Newton's method for linear and nonlinear equations and unconstrained optimization. Global methods, nonlinear least squares, integral equations.





## Math 693b: Informal Prerequisites

`Math 531` and (`Math 541` or `Math 542` or `Math 543` or `Math 693a`) and `Mathematical Software (e.g. matlab)`

Essential knowledge of PDEs, some experience with "mathematical programming" in some language (*e.g.* `matlab`), and linear algebra.

Knowledge of **Fourier, Real, and Complex analysis** is not required, but incredibly useful!

**If you don't know how to write code, this class will be VERY PAINFUL.**

## Possibilies: Finite Element Methods and/or Mimetic Finite Difference Schemes

This class will primarily focus on **Finite Difference Methods**.

We will spend some time discussing **Finite Element Methods** and/or **Mimetic Finite Difference Methods**, and possibly **Spectral Methods** in the latter part of the semester.

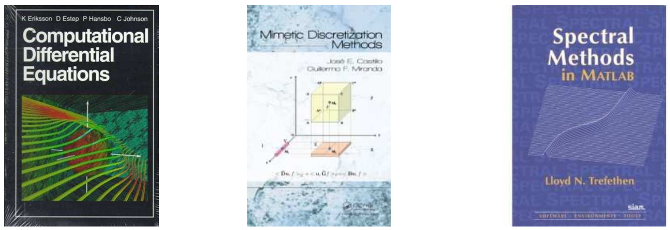


**Figure:** "Computational Differential Equations." K. Eriksson, D. J. Estep, P. Hansbo, and C. Johnson. (Cambridge University Press, 1996); "Mimetic Discretization Methods." J. E. Castillo, and G. F. Miranda. (CRC Press, 2013); "Spectral Methods in MATLAB." L. N. Trefethen (SIAM, 2000).





## Some PDE Models 1 of 5

### The Heat Equation

$$T_t - \kappa(\bar{\mathbf{x}})\nabla^2 T = f(\bar{\mathbf{x}}, t)$$

The heat equation describes heat transfer in a medium. $\kappa$ is the thermal diffusivity and $T$ the temperature. (heat.mpg, hmovie2d-ic6.avi)

### The Wave Equation

$$\frac{1}{c(\bar{\mathbf{x}})^2}\Phi_{tt} - \nabla^2\Phi = f(\bar{\mathbf{x}}, t)$$

The wave equation describes propagation of waves with (location-dependent) speed $c(\bar{\mathbf{x}})$. (wmovie2d-ic3.avi)





## Some PDE Models 2 of 5

### The Schrödinger Equation

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} - \left[\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - V(x)\right]\Psi(x,t) = 0$$

The Schrödinger equation (here in time-dependent one-dimensional form) is the fundamental equation of physics for describing **quantum mechanical behavior.** It is also often called the Schrödinger wave equation, and is a partial differential equation that describes how the wavefunction $[\Psi(x,t)]$ of a physical system evolves over time.

$\hbar$ is Planck's constant divided by $2\pi$, $V(x)$ a potential, and $i$ the imaginary unit $\sqrt{-1}$. (smovie-fn7.avi)





## Some PDE Models 3 of 5

### The Korteweg-de Vries Equation

$$\frac{\partial\eta(x,t)}{\partial t} = \frac{3}{2}\sqrt{\frac{g}{h}}\left(\eta(x,t)\frac{\partial\eta(x,t)}{\partial x} + \frac{2}{3}\frac{\partial\eta(x,t)}{\partial x} + \frac{1}{3}\sigma\frac{\partial^3\eta(x,t)}{\partial x^3}\right)$$

with $\sigma = h^3/3 - Th/(g\rho)$.

The Korteweg-de Vries equation governs **weakly nonlinear shallow waves**. $h$ is the channel height, $T$ is the surface tension, $g$ the gravitational acceleration and $\rho$ the density.

The more commonly seen nondimensionalized version of the KdV equation takes the form

$$u_t + u_{xxx} - 6uu_x = 0$$

(kdv-2soliton.mpg, kdv-sin.mpg)

## Some PDE Models



### The Fokker-Planck Equation

$$\frac{\partial}{\partial t} G(\bar{\mathbf{r}}, \bar{\mathbf{v}}; \bar{\mathbf{r}}_0, \bar{\mathbf{v}}_o; t) + \bar{\mathbf{v}} \cdot \nabla_{\bar{\mathbf{r}}} G(\bar{\mathbf{r}}, \bar{\mathbf{v}}; \bar{\mathbf{r}}_0, \bar{\mathbf{v}}_o; t) =$$

$$\nabla_{\bar{\mathbf{v}}} \cdot \xi \bar{\mathbf{v}} G(\bar{\mathbf{r}}, \bar{\mathbf{v}}; \bar{\mathbf{r}}_0, \bar{\mathbf{v}}_o; t) + \nabla^2_{\bar{\mathbf{v}}} \frac{\xi k T}{m} G(\bar{\mathbf{r}}, \bar{\mathbf{v}}; \bar{\mathbf{r}}_0, \bar{\mathbf{v}}_o; t)$$

The Fokker-Planck equation describes **stochastic evolution**, describing drift and diffusion of a density function. $G$ is the probability density; $\bar{\mathbf{r}}$ and $\bar{\mathbf{r}}_0$ positions; $\bar{\mathbf{v}}$ and $\bar{\mathbf{v}}_0$ velocities.

(fokker-planck-dark-matter.avi, fokker-planck-landau.mpeg)

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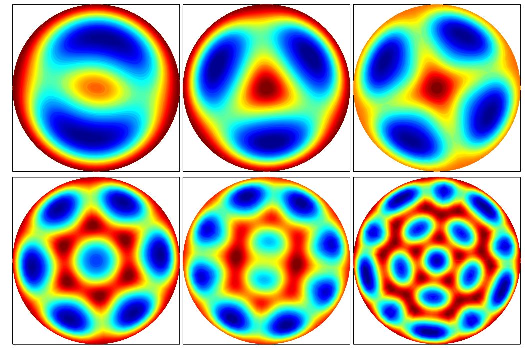
## Some PDE Models



### The Kuramoto-Sivashinsky Equation

$$u_t + \nabla^4 u + \nabla^2 u + \frac{1}{2}|\nabla u|^2 = 0$$

The Kuramoto-Sivashinsky equation describes the pattern formation of cellular flames stabilized on a circular porous plug burner.



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## Questions, Comments, Administrative Stuff...

**1/30** Last day to add/drop classes; Last day to add classes; or change grading basis. No schedule adjustments allowed after 11:59 p.m. on this date.

**1/30** Last day to file application for bachelors degree or advanced degree for May and August 2018 graduation.

**3/23** Final day for submitting thesis (without risk) to Montezuma Publishing for thesis review to ensure graduation in May 2018.

**Questions?**

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