Numerical Solutions to PDEs

Lecture Notes #1 — Introduction

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San Diego State University

Outline

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Academic Life — MSc

- KTH — MSc. Engineering Physics, Royal Institute of Technology (KTH), Stockholm, Sweden. Thesis Advisers: Michael Benedicks, Department of Mathematics KTH, and Erik Aurell, Stockholm University, Department of Mathematics. Thesis Topic: *"A Renormalization Technique for Families with Flat Maxima."*

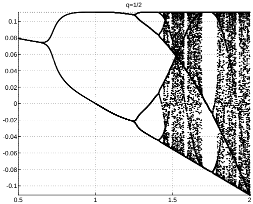

Figure: Bifurcation diagram for the family $f_{a,\frac{1}{2}}$ [BLOMGREN-1994]

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Academic Life — PhD

- PhD. UCLA Department of Mathematics. Adviser: Tony F. Chan. PDE-Based Methods for Image Processing. Thesis title: *"Total Variation Methods for Restoration of Vector Valued Images."*

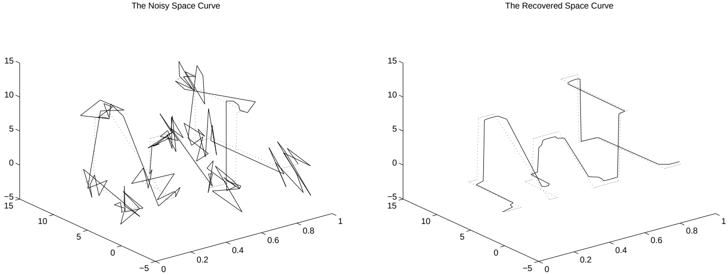

Figure: The noisy (SNR = 4.62 dB), and recovered space curves. Notice how the edges are recovered. [BLOMGREN-1998]

Academic Life — Postdoc

- Research Associate. Stanford University, Department of Mathematics. Main Focus: Time Reversal and Imaging in Random Media (with George Papanicolaou, *et. al.*)

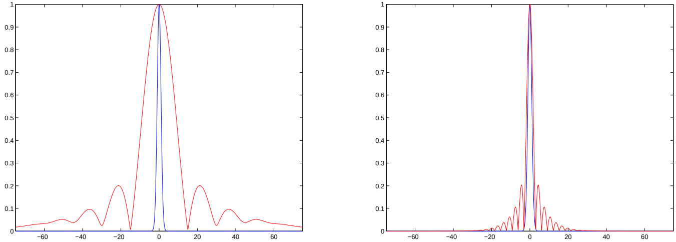

Figure: Comparison of the theoretical formula for a medium with $L = 600\,m$, $a_e = 195\,m$, $\gamma = 2.12 \times 10^{-5}\,m^{-1}$. [LEFT] shows a homogeneous medium, $\gamma = 0$, with $a = 40\,m$ TRM (in red / wide Fresnel zone), and a random medium with $\gamma = 2.12 \times 10^{-5}$ (in blue). [RIGHT] shows $\gamma = 0$, with $a = a_e = 195\,m$ (in red), and $\gamma = 2.12 \times 10^{-5}$, with $a = 40\,m$ (in blue). The match confirms the validity of [the theory]. [BLOMGREN-PAPANICOLAOU-ZHAO-2002]

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Academic Life — Professor

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- Professor, SDSU, Department of Mathematics and Statistics. Projects: Computational Combustion, Biomedical Imaging (Mitochondrial Structures, Heartcell Contractility, Skin/Prostate Cancer Classification), carbon sequestration, compressed sensing.

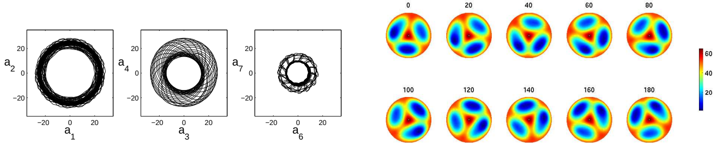

Figure: [LEFT] Phase-space projections produced by the time coefficients of the POD decomposition of the rotating pattern shown in [RIGHT]. [BLOMGREN-GASNER-PALACIOS-2005]

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Contact Information

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Office Hours	TuTh 11:00am – 12:00pm, W 1:00pm – 2:30pm and by appointment.

Fun Times... ⇒ Endurance Sports

- Triathlons:
  - (13) Ironman distance (2.4 + 112 + 26.2) — 11:48:57 [PR]
  - (16) Half Ironman distance — 5:14:20
- Running
  - (1) 100k Race (62.1 miles) — 15:37:46
  - (1) Trail Double-marathon (52 miles) — 10:59:00
  - (5) Trail 50-mile races — 9:08:46
  - (8) Trail 50k (31 mile) races — 5:20:57
  - (15) Road/Trail Marathons — 3:26:19 (7:52/mi)
  - (28) Road/Trail Half Marathons — 1:36:25 (7:21/mi)

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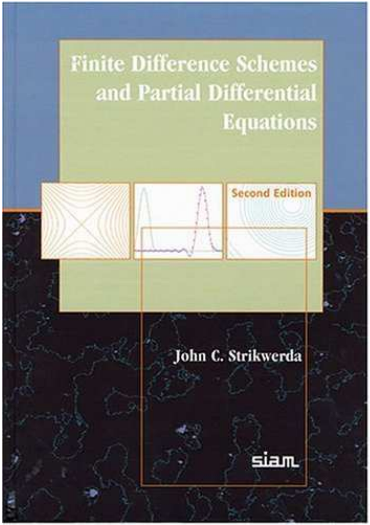
Math 693b: Literature



"Finite Difference Schemes and Partial Differential Equations," 2nd Edition.

Author: John C. Strikwerda.

Publisher: Society for Industrial and Applied Mathematics.

ISBN: **0-89871-567-9**.

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Class notes and web-page.
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Syllabus

Chapter	Title
1	**Hyperbolic Partial Differential Equations**
2	Analysis of Finite Difference Schemes
3	Order of Accuracy of Finite Difference Schemes
4	Stability for Multistep Methods
5	Dissipation and Dispersion
6	**Parabolic Partial Differential Equations**
7	Systems of PDEs in Higher Dimensions
8	Second-Order Equations
9	Analysis of Well-Posed and Stable Problems
10	Convergence Estimates for Initial Value Problems
11	Well-Posed and Stable Initial-Boundary Value Problems
12	**Elliptic Partial Differential Equations** and Difference Schemes
13	Linear Iterative Methods
14	Steepest Descent and Conjugate Gradient Methods.

Grading

Homework*	50%
Project$^\times$	50%

* $\approx$ 7 assignments; first $\approx$ 3 mostly theoretical with some computational components, last $\approx$ 4 "purely" computational.

$\times$ Details to be discussed.

Expectations and Procedures, I

- Most class attendance is "OPTIONAL" — Homework and announcements will be posted on the class web page. If/when you attend class:
  - Please be on time.
  - Please pay attention.
  - Please turn off mobile phones.
  - Please be courteous to other students and the instructor.
  - Abide by university statutes, and all applicable local, state, and federal laws.

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Expectations and Procedures, II

- Please, turn in assignments on time. (The instructor reserves the right not to accept late assignments.)
- The instructor will make special arrangements for students with documented learning disabilities and will `try` to make accommodations for other unforeseen circumstances, *e.g.* illness, personal/family crises, etc. in a way that is fair to all students enrolled in the class. **Please contact the instructor EARLY regarding special circumstances.**
- Students are expected **and encouraged** to ask questions in class!
- Students are expected **and encouraged** to to make use of office hours! If you cannot make it to the scheduled office hours: contact the instructor to schedule an appointment!

Expectations and Procedures, II $\frac{I}{II}$

Late HW Policy

- 10% loss of value / full week late. Examples:
  - 6 days 23 hrs 59 minutes 59 seconds late = FULL VALUE
  - 7 days 0 hrs 0 min 1 sec late = 10% LOSS OF VALUE
  - 14 days 0 hrs 0 min 1 sec late = 20% LOSS OF VALUE

Expectations and Procedures, III

- Missed midterm exams: Don't miss exams! The instructor reserves the right to schedule make-up exams, make such exams oral presentation, and/or base the grade solely on other work (including the final exam).
- Missed final exam: Don't miss the final! Contact the instructor ASAP or a grade of incomplete or F will be assigned.
- **Academic honesty**: submit your own work — but feel free to discuss homework with other students in the class!

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Honesty Pledges, I

- The following **Honesty Pledge** must be included in all programs you submit (as part of homework and/or projects):
  - I, (your name), pledge that this program is completely my own work, and that I did not take, borrow or steal code from any other person, and that I did not allow any other person to use, have, borrow or steal portions of my code. I understand that if I violate this honesty pledge, I am subject to disciplinary action pursuant to the appropriate sections of the San Diego State University Policies.
- Work missing the honesty pledge **may not be graded!**

Honesty Pledges, II

- Larger reports must contain the following text:
  - I, (your name), pledge that this report is completely my own work, and that I did not take, borrow or steal any portions from any other person. Any and all references I used are clearly cited in the text. I understand that if I violate this honesty pledge, I am subject to disciplinary action pursuant to the appropriate sections of the San Diego State University Policies. `Your signature`.
- Work missing the honesty pledge `may not be graded!`

Math 693b: Computer Resources

You need access to a computing environment in which to write your code; — you may want to use a combination of Matlab (for quick prototyping and short homework assignments) and C/C++ or Fortran (or something completely different).

Free C/C++ (gcc) and Fortran (f77) compilers are available for Linux/UNIX.

You may also want to consider buying the student version of Matlab: `http://www.mathworks.com/`

SDSU students can download a copy of matlab from
http://edoras.sdsu.edu/∼download/matlab.html

[Licensing Subject to Change With Minimal Notice]

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Math 693b: Formal Prerequisites (Graduate Bulletin)

Math 531, Math 537 and Math 693a

531 ⇒ **PDEs**

- Boundary value problems for the heat and wave equations: eigenfunction expansions, Sturm-Liouville theory and Fourier series. D'Alembert's solution to wave equation; characteristics. Laplace's equation, maximum principles, Bessel functions.

537 ⇒ **ODEs**

- Theory of ODEs; existence and uniqueness, dependence on initial conditions and parameters, linear systems, stability and asymptotic behavior, plane autonomous systems, series solutions at regular singular points.

693a ⇒ **Advanced Numerical Analysis (Numerical Optimization)**

- Numerical optimization, Newton's method for linear and nonlinear equations and unconstrained optimization. Global methods, nonlinear least squares, integral equations.

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Math 693b: Informal Prerequisites

`Math 531` and (`Math 541` or `Math 542` or `Math 543` or `Math 693a`) and `Mathematical Software` (*e.g.* `matlab`)

Essential knowledge of PDEs, some experience with "mathematical programming" in some language (*e.g.* `matlab`), and linear algebra.

Knowledge of **Fourier, Real, and Complex analysis** is not required, but incredibly useful!

If you don't know how to write code, this class will be VERY PAINFUL.

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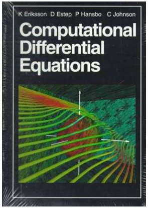
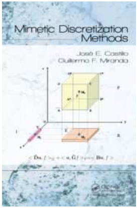
Possibilies: Finite Element Methods and/or Mimetic Finite Difference Schemes

This class will primarily focus on **Finite Difference Methods**.

We will spend some time discussing **Finite Element Methods** and/or **Mimetic Finite Difference Methods**, and possibly **Spectral Methods** in the latter part of the semester.





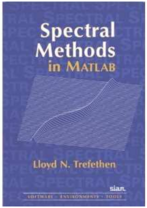


Figure: "Computational Differential Equations." K. Eriksson, D. J. Estep, P. Hansbo, and C. Johnson. (Cambridge University Press, 1996); "Mimetic Discretization Methods." J. E. Castillo, and G. F. Miranda. (CRC Press, 2013); "Spectral Methods in MATLAB." L. N. Trefethen (SIAM, 2000).

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Some PDE Models

The Heat Equation

$$T_t - \kappa(\bar{\mathbf{x}})\nabla^2 T = f(\bar{\mathbf{x}}, t)$$

The heat equation describes heat transfer in a medium. κ is the thermal diffusivity and T the temperature. (heat.mpg, hmovie2d-ic6.avi)

The Wave Equation

$$\frac{1}{c(\bar{\mathbf{x}})^2}\Phi_{tt} - \nabla^2\Phi = f(\bar{\mathbf{x}}, t)$$

The wave equation describes propagation of waves with (location-dependent) speed $c(\bar{\mathbf{x}})$. (wmovie2d-ic3.avi)

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Some PDE Models

The Schrödinger Equation

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} - \left[\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - V(x)\right]\Psi(x,t) = 0$$

The Schrödinger equation (here in time-dependent one-dimensional form) is the fundamental equation of physics for describing **quantum mechanical behavior.** It is also often called the Schrödinger wave equation, and is a partial differential equation that describes how the wavefunction $[\Psi(x,t)]$ of a physical system evolves over time.

$\hbar$ is Planck's constant divided by 2π, $V(x)$ a potential, and i the imaginary unit $\sqrt{-1}$. (smovie-fn7.avi)

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The Korteweg-de Vries Equation

$$\frac{\partial \eta(x,t)}{\partial t} = \frac{3}{2}\sqrt{\frac{g}{h}}\left(\eta(x,t)\frac{\partial \eta(x,t)}{\partial x} + \frac{2}{3}\frac{\partial \eta(x,t)}{\partial x} + \frac{1}{3}\sigma\frac{\partial^3 \eta(x,t)}{\partial x^3}\right)$$

with $\sigma = h^3/3 - Th/(g\rho)$.

The Korteweg-de Vries equation governs **weakly nonlinear shallow waves**. h is the channel height, T is the surface tension, g the gravitational acceleration and ρ the density.

The more commonly seen nondimensionalized version of the KdV equation takes the form

$$u_t + u_{xxx} - 6uu_x = 0$$

(kdv-2soliton.mpg, kdv-sin.mpg)

Some PDE Models

The Fokker-Planck Equation

$$\frac{\partial}{\partial t} G(\bar{\mathbf{r}}, \bar{\mathbf{v}}; \bar{\mathbf{r}}_0, \bar{\mathbf{v}}_o; t) + \bar{\mathbf{v}} \cdot \nabla_{\bar{\mathbf{r}}} G(\bar{\mathbf{r}}, \bar{\mathbf{v}}; \bar{\mathbf{r}}_0, \bar{\mathbf{v}}_o; t) =$$
$$\nabla_{\bar{\mathbf{v}}} \cdot \xi \bar{\mathbf{v}} G(\bar{\mathbf{r}}, \bar{\mathbf{v}}; \bar{\mathbf{r}}_0, \bar{\mathbf{v}}_o; t) + \nabla^2_{\bar{\mathbf{v}}} \frac{\xi k T}{m} G(\bar{\mathbf{r}}, \bar{\mathbf{v}}; \bar{\mathbf{r}}_0, \bar{\mathbf{v}}_o; t)$$

The Fokker-Planck equation describes **stochastic evolution**, describing drift and diffusion of a density function. G is the probability density; $\bar{\mathbf{r}}$ and $\bar{\mathbf{r}}_0$ positions; $\bar{\mathbf{v}}$ and $\bar{\mathbf{v}}_0$ velocities.

(**fokker-planck-dark-matter.avi, fokker-planck-landau.mpeg**)

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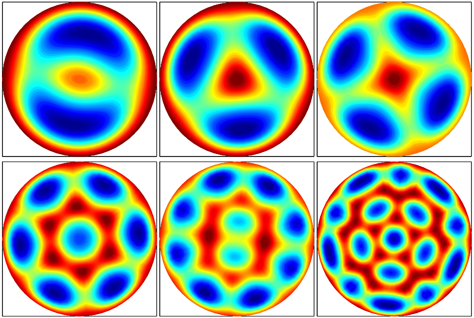
Some PDE Models

The Kuramoto-Sivashinsky Equation

$$u_t + \nabla^4 u + \nabla^2 u + \frac{1}{2}|\nabla u|^2 = 0$$

The Kuramoto-Sivashinsky equation describes the pattern formation of cellular flames stabilized on a circular porous plug burner.



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Questions, Comments, Administrative Stuff...

1/30 Last day to add/drop classes; Last day to add classes; or change grading basis. No schedule adjustments allowed after 11:59 p.m. on this date.

1/30 Last day to file application for bachelors degree or advanced degree for May and August 2018 graduation.

3/23 Final day for submitting thesis (without risk) to Montezuma Publishing for thesis review to ensure graduation in May 2018.

Questions?