

Numerical Solutions to PDEs

Lecture Notes #1 — Introduction

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Outline

- 1 **The Professor**
 - Academic Life
 - Contact Information, Office Hours
 - Non-Academic Life
- 2 **The Class — Overview**
 - Literature & Syllabus
 - Grading
 - Expectations and Procedures
- 3 **The Class...**
 - Resources
 - Formal Prerequisites
- 4 **Introduction**
 - Some PDE Models



KTH

- MSc. Engineering Physics, Royal Institute of Technology (KTH), Stockholm, Sweden. Thesis Advisers: Michael Benedicks, Department of Mathematics KTH, and Erik Aurell, Stockholm University, Department of Mathematics. Thesis Topic: “A Renormalization Technique for Families with Flat Maxima.”

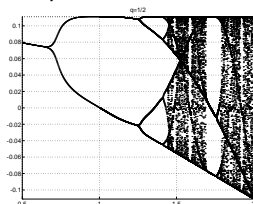
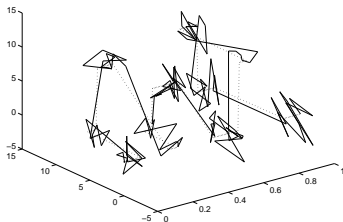


Figure: Bifurcation diagram for the family $f_{a, \frac{1}{2}}$ [BLOMGREN-1994]



- UCLA PhD. UCLA Department of Mathematics. Adviser: Tony F. Chan. PDE-Based Methods for Image Processing. Thesis title: *“Total Variation Methods for Restoration of Vector Valued Images.”*

The Noisy Space Curve



The Recovered Space Curve

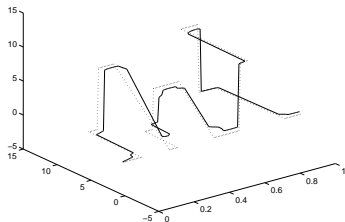


Figure: The noisy (SNR = 4.62 dB), and recovered space curves. Notice how the edges are recovered. [BLOMGREN-1998]



- Research Associate. Stanford University, Department of Mathematics. Main Focus: Time Reversal and Imaging in Random Media (with George Papanicolaou, *et. al.*)

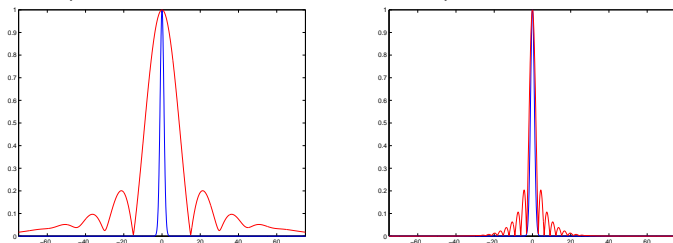


Figure: Comparison of the theoretical formula for a medium with $L = 600 \text{ m}$, $a_e = 195 \text{ m}$, $\gamma = 2.12 \times 10^{-5} \text{ m}^{-1}$. [LEFT] shows a homogeneous medium, $\gamma = 0$, with $a = 40 \text{ m}$ TRM (in red / wide Fresnel zone), and a random medium with $\gamma = 2.12 \times 10^{-5}$ (in blue). [RIGHT] shows $\gamma = 0$, with $a = a_e = 195 \text{ m}$ (in red), and $\gamma = 2.12 \times 10^{-5}$, with $a = 40 \text{ m}$ (in blue). The match confirms the validity of [the theory]. [BLOMGREN-PAPANICOLAOU-ZHAO-2009]





SAN DIEGO STATE
UNIVERSITY

- Professor, SDSU, Department of Mathematics and Statistics. Projects: Computational Combustion, Biomedical Imaging (Mitochondrial Structures, Heartcell Contractility, Skin/Prostate Cancer Classification), carbon sequestration, compressed sensing.

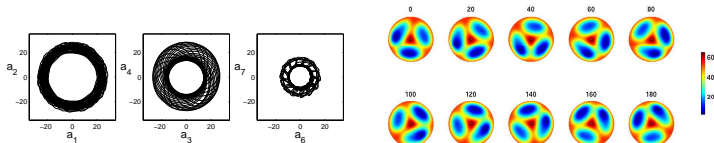


Figure: [LEFT] Phase-space projections produced by the time coefficients of the POD decomposition of the rotating pattern shown in [RIGHT]. [BLOMGREN-GASNER-PALACIOS-2005]



SAN DIEGO STATE
UNIVERSITY

Contact Information



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Email	blomgren.peter@gmail.com
Web	http://terminus.sdsu.edu/SDSU/Math693b/
Office Hours	TuTh 11:00am – 12:00pm, W 1:00pm – 2:30pm and by appointment.

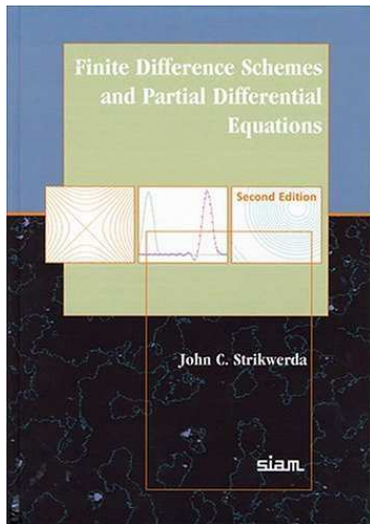


Fun Times... ⇒ Endurance Sports



- Triathlons:
 - (13) Ironman distance (2.4 + 112 + 26.2) — 11:48:57 [PR]
 - (16) Half Ironman distance — 5:14:20
- Running
 - (1) 100k Race (62.1 miles) — 15:37:46
 - (1) Trail Double-marathon (52 miles) — 10:59:00
 - (5) Trail 50-mile races — 9:08:46
 - (8) Trail 50k (31 mile) races — 5:20:57
 - (15) Road/Trail Marathons — 3:26:19 (7:52/mi)
 - (28) Road/Trail Half Marathons — 1:36:25 (7:21/mi)

Math 693b: Literature



“Finite Difference Schemes and Partial Differential Equations,” **2nd Edition**.

Author: John C. Strikwerda.

Publisher: Society for Industrial and Applied Mathematics.

ISBN: **0-89871-567-9**.

Class notes and web-page.



Syllabus

Chapter	Title
1	Hyperbolic Partial Differential Equations
2	Analysis of Finite Difference Schemes
3	Order of Accuracy of Finite Difference Schemes
4	Stability for Multistep Methods
5	Dissipation and Dispersion
6	Parabolic Partial Differential Equations
7	Systems of PDEs in Higher Dimensions
8	Second-Order Equations
9	Analysis of Well-Posed and Stable Problems
10	Convergence Estimates for Initial Value Problems
11	Well-Posed and Stable Initial-Boundary Value Problems
12	Elliptic Partial Differential Equations and Difference Schemes
13	Linear Iterative Methods
14	Steepest Descent and Conjugate Gradient Methods.



Grading

Homework*	50%
Project [×]	50%

- * \approx 7 assignments; first \approx 3 mostly theoretical with some computational components, last \approx 4 “purely” computational.
- × Details to be discussed.

Expectations and Procedures, I

- Most class attendance is “OPTIONAL” — Homework and announcements will be posted on the class web page. If/when you attend class:

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 - Please be on time.
 - Please pay attention.
 - Please turn off mobile phones.
 - Please be courteous to other students and the instructor.
 - Abide by university statutes, and all applicable local, state, and federal laws.



Expectations and Procedures, II

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- Students are expected **and encouraged** to ask questions in class!
- Students are expected **and encouraged** to to make use of office hours! If you cannot make it to the scheduled office hours: contact the instructor to schedule an appointment!

Late HW Policy

- 10% loss of value / full week late. Examples:
 - 6 days 23 hrs 59 minutes 59 seconds late = FULL VALUE
 - 7 days 0 hrs 0 min 1 sec late = 10% LOSS OF VALUE
 - 14 days 0 hrs 0 min 1 sec late = 20% LOSS OF VALUE

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- Missed final exam: Don't miss the final! Contact the instructor ASAP or a grade of incomplete or F will be assigned.
- **Academic honesty**: submit your own work — but feel free to discuss homework with other students in the class!

Honesty Pledges, I

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- Work missing the honesty pledge **may not be graded!**

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Math 693b: Computer Resources

You need access to a computing environment in which to write your code; — you may want to use a combination of Matlab (for quick prototyping and short homework assignments) and C/C++ or Fortran (or something completely different).

Free C/C++ (gcc) and Fortran (f77) compilers are available for Linux/UNIX.

You may also want to consider buying the student version of Matlab:
<http://www.mathworks.com/>

SDSU students can download a copy of matlab from

<http://edoras.sdsu.edu/~download/matlab.html>

[LICENSING SUBJECT TO CHANGE WITH MINIMAL NOTICE]



Math 531, Math 537 and Math 693a

531 \Rightarrow **PDEs**

- Boundary value problems for the heat and wave equations: eigenfunction expansions, Sturm-Liouville theory and Fourier series. D'Alembert's solution to wave equation; characteristics. Laplace's equation, maximum principles, Bessel functions.

537 \Rightarrow **ODEs**

- Theory of ODEs; existence and uniqueness, dependence on initial conditions and parameters, linear systems, stability and asymptotic behavior, plane autonomous systems, series solutions at regular singular points.

693a \Rightarrow **Advanced Numerical Analysis (Numerical Optimization)**

- Numerical optimization, Newton's method for linear and nonlinear equations and unconstrained optimization. Global methods, nonlinear least squares, integral equations.

Math 693b: Informal Prerequisites

Math 531 and (Math 541 or Math 542 or Math 543 or Math 693a)
and Mathematical Software (*e.g.* matlab)

Essential knowledge of PDEs, some experience with “mathematical programming” in some language (*e.g.* matlab), and linear algebra.

Knowledge of **Fourier, Real, and Complex analysis** is not required, but incredibly useful!

**If you don't know how to write code,
this class will be VERY PAINFUL.**

Possibilities: Finite Element Methods and/or Mimetic Finite Difference Schemes

This class will primarily focus on **Finite Difference Methods**. We will spend some time discussing **Finite Element Methods** and/or **Mimetic Finite Difference Methods**, and possibly **Spectral Methods** in the latter part of the semester.

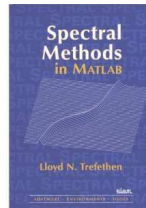
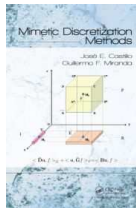
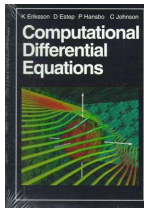


Figure: “Computational Differential Equations.” K. Eriksson, D. J. Estep, P. Hansbo, and C. Johnson. (Cambridge University Press, 1996); “Mimetic Discretization Methods.” J. E. Castillo, and G. F. Miranda. (CRC Press, 2013); “Spectral Methods in MATLAB.” L. N. Trefethen (SIAM, 2000).

The Heat Equation

$$T_t - \kappa(\bar{\mathbf{x}})\nabla^2 T = f(\bar{\mathbf{x}}, t)$$

The heat equation describes heat transfer in a medium. κ is the thermal diffusivity and T the temperature. ([heat.mpg](#), [hmovie2d-ic6.avi](#))

The Wave Equation

$$\frac{1}{c(\bar{\mathbf{x}})^2}\Phi_{tt} - \nabla^2\Phi = f(\bar{\mathbf{x}}, t)$$

The wave equation describes propagation of waves with (location-dependent) speed $c(\bar{\mathbf{x}})$. ([wmovie2d-ic3.avi](#))

The Schrödinger Equation

$$i\hbar \frac{\partial \Psi(x, t)}{\partial t} - \left[\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - V(x) \right] \Psi(x, t) = 0$$

The Schrödinger equation (here in time-dependent one-dimensional form) is the fundamental equation of physics for describing **quantum mechanical behavior**. It is also often called the Schrödinger wave equation, and is a partial differential equation that describes how the wavefunction $[\Psi(x, t)]$ of a physical system evolves over time.

\hbar is Planck's constant divided by 2π , $V(x)$ a potential, and i the imaginary unit $\sqrt{-1}$. ([smovie-fn7.avi](#))



The Korteweg-de Vries Equation

$$\frac{\partial \eta(x, t)}{\partial t} = \frac{3}{2} \sqrt{\frac{g}{h}} \left(\eta(x, t) \frac{\partial \eta(x, t)}{\partial x} + \frac{2}{3} \frac{\partial \eta(x, t)}{\partial x} + \frac{1}{3} \sigma \frac{\partial^3 \eta(x, t)}{\partial x^3} \right)$$

with $\sigma = h^3/3 - Th/(g\rho)$.

The Korteweg-de Vries equation governs **weakly nonlinear shallow waves**. h is the channel height, T is the surface tension, g the gravitational acceleration and ρ the density.

The more commonly seen nondimensionalized version of the KdV equation takes the form

$$u_t + u_{xxx} - 6uu_x = 0$$

The Fokker-Planck Equation

$$\frac{\partial}{\partial t} G(\bar{\mathbf{r}}, \bar{\mathbf{v}}; \bar{\mathbf{r}}_0, \bar{\mathbf{v}}_0; t) + \bar{\mathbf{v}} \cdot \nabla_{\bar{\mathbf{r}}} G(\bar{\mathbf{r}}, \bar{\mathbf{v}}; \bar{\mathbf{r}}_0, \bar{\mathbf{v}}_0; t) = \\ \nabla_{\bar{\mathbf{v}}} \cdot \xi \bar{\mathbf{v}} G(\bar{\mathbf{r}}, \bar{\mathbf{v}}; \bar{\mathbf{r}}_0, \bar{\mathbf{v}}_0; t) + \nabla_{\bar{\mathbf{v}}}^2 \frac{\xi kT}{m} G(\bar{\mathbf{r}}, \bar{\mathbf{v}}; \bar{\mathbf{r}}_0, \bar{\mathbf{v}}_0; t)$$

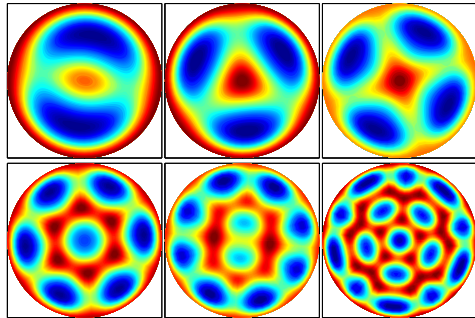
The Fokker-Planck equation describes **stochastic evolution**, describing drift and diffusion of a density function. G is the probability density; $\bar{\mathbf{r}}$ and $\bar{\mathbf{r}}_0$ positions; $\bar{\mathbf{v}}$ and $\bar{\mathbf{v}}_0$ velocities.

([fokker-planck-dark-matter.avi](#), [fokker-planck-landau.mpeg](#))

The Kuramoto-Sivashinsky Equation

$$u_t + \nabla^4 u + \nabla^2 u + \frac{1}{2} |\nabla u|^2 = 0$$

The Kuramoto-Sivashinsky equation describes the pattern formation of cellular flames stabilized on a circular porous plug burner.



Questions, Comments, Administrative Stuff...

- 1/30** Last day to add/drop classes; Last day to add classes; or change grading basis. No schedule adjustments allowed after 11:59 p.m. on this date.
- 1/30** Last day to file application for bachelors degree or advanced degree for May and August 2018 graduation.
- 3/23** Final day for submitting thesis (without risk) to Montezuma Publishing for thesis review to ensure graduation in May 2018.

Questions?