

# Numerical Solutions to PDEs

Lecture Notes #4

— Analysis of Finite Difference Schemes —  
Fourier Analysis; Von Neumann Analysis

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**Convergence:** The desired result; as we refine the grid, the numerical solution of the Finite Difference Scheme (FDS) should better and better represent the exact (continuous) solution of the PDE.

**Consistency:** Easily checked by **Taylor expansion** — the expansion of the FDS should give the PDE + terms that go to zero as  $(h, k) \rightarrow 0$ .

**Stability:** An  $\ell_2$ -energy bound on the solution of the FDS in terms of the initial condition (+ further levels of initialization for multi-step schemes). Hard to check using the definitions — we start developing tools today!

**Well-Posedness:** A property of the PDE for IVPs — An  $L^2$ -energy bound on the solution in terms of the initial conditions.



## The CFL-condition

Courant-Friedrichs-Lewy condition  $|a\lambda| \leq 1$  (for explicit one-step schemes applied to  $u_t + au_x + bu = f$ ) is **necessary** (but not sufficient) for stability. It expresses the need for the numerical speed of propagation  $\lambda^{-1}$  to match or exceed the physical speed of propagation  $a$ .

## Theorem (The Lax-Richtmyer Equivalence Theorem)

*A consistent finite difference scheme for a partial differential equation for which the initial value problem is well-posed is convergent **if and only if** it is stable.*

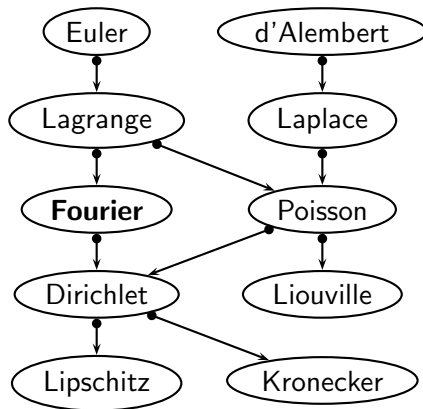
## A Crash Course in Fourier Analysis



**Figure:** Jean-Baptiste Joseph Fourier (21 March 1768 – 16 May 1830). Advisor: Joseph Louis Lagrange. Student: Gustav Peter Lejeune Dirichlet (+1).

## Truncated Genealogy

(Advisor  $\rightarrow$  Student)



## A Crash Course in Fourier Analysis

The **Fourier transform** of a function  $u(x)$ ,  $x \in \mathbb{R}$  is defined by

$$\hat{u}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} u(x) dx.$$

The **Fourier inversion formula**

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \hat{u}(\omega) d\omega,$$

recovers the function from its Fourier transform.

Essentially, the Fourier transform representation expresses  $u(x)$  as an infinite superposition of (complex) waves  $e^{i\omega x} = \cos(\omega x) + i \sin(\omega x)$ , with amplitudes  $\hat{u}(\omega)$ .

(!)  $u(x)$  and  $\hat{u}(\omega)$  must satisfy certain criteria for the integrals (above) to be well-defined. We sweep those details under the rug, and refer to **Math 668: Applied Fourier Analysis**.

## Example

With Correction of Typo (Strikwerda p.38)

We consider the function

$$u(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0. \end{cases}$$

The Fourier transform of  $u(x)$  is given by

$$\underbrace{\widehat{u}(\omega)}_{\text{correction}} = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-i\omega x} e^{-x} dx = \frac{1}{\sqrt{2\pi}} \frac{1}{1 + i\omega}.$$

The tools needed for evaluation of such integrals can be found in **Math 532 + 631A + 631B: Complex Analysis**.

Tables of Fourier transforms can be found online in various places; ask uncle Google for guidance.



## Fourier Transform Tables: A Warning

There are several ways of defining the Fourier transform — the normalization constants for the forward and inverse transforms are chosen from one of the following three set

$$\left\{ \frac{\mathbf{1}}{\sqrt{2\pi}}, \frac{\mathbf{1}}{\sqrt{2\pi}} \right\}, \quad \left\{ 1, \frac{1}{2\pi} \right\}, \quad \left\{ \frac{1}{2\pi}, 1 \right\},$$

and the factors in the integrals can be chosen to be

$$\left\{ e^{-i\omega x}, e^{i\omega x} \right\}, \quad \left\{ e^{i\omega x}, e^{-i\omega x} \right\}.$$

For a total of six “natural” ways to define the transform and its inverse. Of course, mathematicians and engineers have agreed to disagree on the definition of the One True Fourier Transform™. — These choices also affect numerical implementations of the discrete Fourier transform...

## Extending the Fourier Transform to Grid Functions, I

For a grid function  $v_m$  defined for all integers coordinates  $m$ , the Fourier transform is given by

$$\hat{v}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-im\xi} v_m,$$

where  $\xi \in [-\pi, \pi]$ , and  $\hat{v}(\pi) = \hat{v}(-\pi)$ .

The inversion formula is given by

$$v_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{im\xi} \hat{v}(\xi) d\xi.$$

## Extending the Fourier Transform to Grid Functions, II

For a grid function  $v_m$  defined for all coordinates  $x_m = h \cdot m$ , the Fourier transform is given by

$$\widehat{v}(\xi) = \frac{h}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-imh\xi} v_m$$

where  $\xi \in [-\pi/h, \pi/h]$ , and  $\widehat{v}(\pi/h) = \widehat{v}(-\pi/h)$ .

The inversion formula is given by

$$v_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi} \widehat{v}(\xi) d\xi.$$

## Parseval's Relations: Preservation of $L^2$ Energy

With the following definition for the  $L^2$  (continuous) energy

$$\|u\|_2 = \sqrt{\int_{-\infty}^{\infty} |u(x)|^2 dx},$$

the following holds

Parseval's Relations

$$\sqrt{\int_{-\infty}^{\infty} |u(x)|^2 dx} = \|u\|_2 = \|\hat{u}\|_2 = \sqrt{\int_{-\infty}^{\infty} |\hat{u}(\omega)|^2 d\omega}$$

$$\sqrt{h \sum_{m=-\infty}^{\infty} |v_m|^2} = \|v\|_2 = \|\hat{v}\|_2 = \sqrt{\int_{-\pi/h}^{\pi/h} |\hat{v}(\xi)|^2 d\xi}$$

These relations are key to our stability analysis, and are also a big reason why measuring quantities in the  $L^2$  (and/or  $\ell_2$ ) norm is usually a Good Thing™ — many times the norm expresses a natural physical energy, *and* that energy is preserved under the Fourier transform.



## The Road to Stability: Using Parseval's Relations

Using Parseval's relations, we can rewrite the inequalities that appeared in the definition of stability (last lecture)

$$h \sum_{m=-\infty}^{\infty} |v_m^n|^2 \leq C_T h \sum_{j=0}^J \sum_{m=-\infty}^{\infty} |v_m^j|^2,$$

and

$$\|v^n\|_h \leq \left[ C_T \sum_{j=0}^J \|v^j\|_h^2 \right]^{1/2} \Leftrightarrow \|v^n\|_h \leq C_T^* \sum_{j=0}^J \|v^j\|_h,$$

by the equivalent inequality (applied in the Fourier domain...)

$$\|\hat{v}^n\|_h \leq C_T^* \sum_{j=0}^J \|\hat{v}^j\|_h.$$

So???

## Fourier Analysis and PDEs

Given the Fourier inversion formula

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \hat{u}(\omega) d\omega,$$

we formally compute the derivative with respect to  $x$ :

$$\frac{\partial u(x)}{\partial x} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} i\omega \hat{u}(\omega) d\omega.$$

This leads us to the stunningly simple, and extremely useful conclusion that

$$\widehat{\left(\frac{\partial u}{\partial x}\right)} = i\omega \hat{u}(\omega)$$

*i.e.* **differentiation** on the physical side, corresponds to **multiplication by  $i\omega$**  on the Fourier transform side.

Higher Derivatives —  $L^2$  — Parseval

It now follows that the squared  $L^2$ -norm of the  $r$ -th derivative is given by

$$\int_{-\infty}^{\infty} \left| \frac{\partial^r u(x)}{\partial x^r} \right|^2 dx = \int_{-\infty}^{\infty} |\omega|^{2r} |\hat{u}(\omega)|^2 d\omega,$$

These quantities exist (*i.e.*  $u$  has  $L^2$  integrable derivatives of order through  $r$ , **if and only if**

$$\int_{-\infty}^{\infty} (1 + |\omega|^2)^r |\hat{u}(\omega)|^2 d\omega < \infty.$$

From this we can define the **function space (Sobolev space)**, also denoted  $W_2^r(\mathbb{R})$ , or  $W^{r,2}(\mathbb{R})$   $H^r(\mathbb{R})$  ( $r > 0$ ) as the set of functions  $u \in L^2(\mathbb{R})$ , for which (note  $H^0(\mathbb{R}) \equiv L^2(\mathbb{R})$ )

$$\|u\|_{H^r} = \sqrt{\int_{-\infty}^{\infty} (1 + |\omega|^2)^r |\hat{u}(\omega)|^2 d\omega} < \infty.$$

## Notations for Norms of Derivatives

We introduce the notation

$$\|D^r u\|^2 = \int_{-\infty}^{\infty} \left| \frac{\partial^r}{\partial x^r} u(x) \right|^2 dx = \int_{-\infty}^{\infty} |\omega|^{2r} |\hat{u}(\omega)|^2 d\omega,$$

and note (for future reference), that the integral over  $x$  is only defined when  $r$  is an integer, but the integral over  $\omega$  can be used for “fractional derivatives.”

OK, lets return to the one-way wave equation...



## Fourier Analysis and the One-Way Wave Equation, I

Consider, with  $u(0, x) = u_0(x)$  specified,

$$u_t + au_x = 0, \quad \Leftrightarrow \quad u_t = -au_x.$$

Fourier transforming in the  $x$ -coordinate, we get

$$\hat{u}_t = -ia\omega\hat{u}, \quad \hat{u}_0(\omega) \text{ given.}$$

This is an Ordinary Differential Equation (ODE) in  $t$ , and the solution is given by

$$\hat{u}(t, \omega) = e^{-ia\omega t} \hat{u}_0(\omega).$$

With the help of the tools we have developed, we can show that this Initial Value Problem is well-posed.

## Fourier Analysis and the One-Way Wave Equation, II

We have, using Parseval's equality

$$\begin{aligned}\int_{-\infty}^{\infty} |u(t, x)|^2 dx &= \int_{-\infty}^{\infty} |\hat{u}(t, \omega)|^2 d\omega = \int_{-\infty}^{\infty} |e^{-i\omega t} \hat{u}_0|^2 d\omega = \\ &= \int_{-\infty}^{\infty} \underbrace{|e^{-i\omega t}|^2}_1 |\hat{u}_0|^2 d\omega = \int_{-\infty}^{\infty} |\hat{u}_0|^2 d\omega = \int_{-\infty}^{\infty} |u_0|^2 dx = \|u_0\|_2^2.\end{aligned}$$

Hence, not only do we have a bound on the energy — we have an exact value, which does not change in time.  $\Rightarrow$  **The IVP is well-posed.**

## Von Neumann Analysis

The application of Fourier analysis which presently is of interest to us is the application to the stability analysis of finite difference schemes; known as **von Neumann analysis**.

Starting from the forward-time-backward-space scheme (suitable only when  $a > 0$ , think about the characteristic) applied to the one-way wave equation ( $u_t + au_x = 0$ ):

$$\frac{v_m^{n+1} - v_m^n}{k} + a \frac{v_m^n - v_{m-1}^n}{h} = 0.$$

We rewrite this in the form ( $\lambda = k/h$ )

$$v_m^{n+1} = (1 - a\lambda)v_m^n + a\lambda v_{m-1}^n.$$

Next we, use the Fourier inversion formula to represent the quantities on the right-hand side....

## Von Neumann Analysis... Moving Along

1 of 2

With

$$v_m^n = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi} \widehat{v}^n(\xi) d\xi,$$

we get

$$v_m^{n+1} = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi} \left[ (1 - a\lambda) + a\lambda \underbrace{e^{-ih\xi}}_{\text{from } v_{m-1}^n} \right] \widehat{v}^n(\xi) d\xi.$$

From the inversion formula we also have

$$v_m^{n+1} = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi} \widehat{v}^{n+1}(\xi) d\xi.$$

We have two representations of the same quantity...

## Von Neumann Analysis... Moving Along

2 of 2

The integrands must be the same, hence

$$\widehat{v}^{n+1}(\xi) = \underbrace{\left[ (1 - a\lambda) + a\lambda e^{-ih\xi} \right]}_{g(h\xi)} \widehat{v}^n(\xi).$$

$g(h\xi)$  is known as the **amplification factor**, and we note that

$$\widehat{v}^n(\xi) = g(h\xi)^n \widehat{v}^0(\xi).$$

If  $|g(h\xi)| > 1$ , then the energy grows exponentially; hence **for stability we must require**  $|g(h\xi)| \leq 1$ .

## Von Neumann Analysis... Closing Out

We let  $\theta = h\xi$ , and use  $e^{-i\theta} = \cos \theta - i \sin \theta$ , and consider  $|g(\theta)|^2$ :

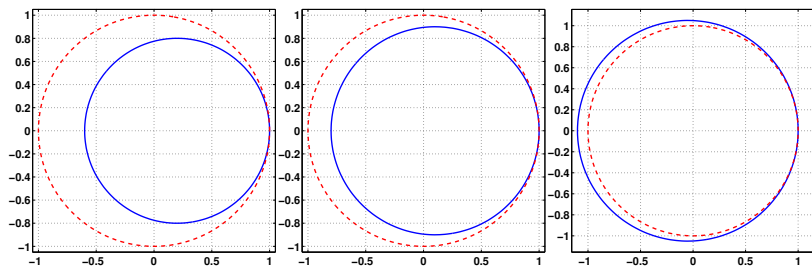
$$\begin{aligned} |g(\theta)|^2 &= (1 - a\lambda + a\lambda \cos \theta)^2 + a^2 \lambda^2 \sin^2 \theta \\ &= \left(1 - 2a\lambda \sin^2 \left(\frac{\theta}{2}\right)\right)^2 + 4a^2 \lambda^2 \sin^2 \left(\frac{\theta}{2}\right) \cos^2 \left(\frac{\theta}{2}\right) \\ &= 1 - 4a\lambda \sin^2 \left(\frac{\theta}{2}\right) + 4a^2 \lambda^2 \sin^4 \left(\frac{\theta}{2}\right) + 4a^2 \lambda^2 \sin^2 \left(\frac{\theta}{2}\right) \cos^2 \left(\frac{\theta}{2}\right) \\ &= 1 - 4a\lambda(1 - a\lambda) \sin^2 \left(\frac{\theta}{2}\right). \end{aligned}$$

Since  $\sin^2 \left(\frac{\theta}{2}\right) \geq 0$ , we must require  $a\lambda \geq 0$  and  $a\lambda \leq 1$  in order for  $|g(\theta)|^2 \leq 1$ . Hence, the scheme is stable for  $\mathbf{0} \leq \mathbf{a}\lambda \leq \mathbf{1}$ .

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$$1 - \cos \theta = 2 \sin^2 \left(\frac{\theta}{2}\right), \quad \sin \theta = 2 \sin \left(\frac{\theta}{2}\right) \cos \left(\frac{\theta}{2}\right)$$

## Von Neumann Analysis: Images of $g(\theta)$



**Figure:** Images of  $g(\theta)$  in the complex plane. For  $a\lambda = 0.8$  and  $a\lambda = 0.9$ , the image (**blue, solid**) is contained in the unit circle (**red, dashed**), but for  $a\lambda = 1.05$  we can clearly see that  $|g(\theta)| > 1$ , except for  $g(0) = 1$ .

With this example in mind, we are ready for the formal criterion for stability.

## Von Neumann Analysis: The Stability Condition

## Theorem (Von Neumann Stability)

A one-step finite difference scheme (with constant coefficients) is stable in a stability region  $\Lambda$  **if and only if** there is a constant  $K$  (independent of  $\theta$ ,  $k$ , and  $h$ ) such that

$$|g(\theta, k, h)| \leq 1 + Kk$$

with  $(k, h) \in \Lambda$ . If  $g(\theta, k, h)$  is independent of  $h$  and  $k$ , the stability condition can be replaced with the restricted stability condition

$$|g(\theta)| \leq 1.$$

Determining stability this way is quite straightforward — only symbolic manipulations of the expression for  $|g(\theta, k, h)|^2$  are needed.



## Example: Forward-Time-Central-Space

The procedure can be stream-lined quite a bit, consider

$$\frac{v_m^{n+1} - v_m^n}{k} + a \frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0.$$

Replace  $v_m^n$  by  $g^n e^{im\theta}$ , and get

$$\begin{aligned} \frac{g^{n+1} e^{im\theta} - g^n e^{im\theta}}{k} + a \frac{g^n e^{i(m+1)\theta} - g^n e^{i(m-1)\theta}}{2h} \\ = g^n e^{im\theta} \left[ \frac{g - 1}{k} + a \frac{e^{i\theta} - e^{-i\theta}}{2h} \right] = 0. \end{aligned}$$

The expression in the square bracket must be zero, and  $e^{i\theta} - e^{-i\theta} = 2i \sin \theta$ , so the amplification factor is given by

$$g(\theta) = 1 - ia\lambda \sin \theta, \quad |g(\theta)|^2 = 1 + (a\lambda)^2 \sin^2 \theta \geq 1.$$

Hence, this scheme is **unstable**.

## Example: Lax-Friedrichs Scheme

The Lax-Friedrichs Scheme is quite similar to FT-CS:

$$\frac{v_m^{n+1} - \frac{1}{2} [v_{m+1}^n + v_{m-1}^n]}{k} + a \frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0$$

Replace  $v_m^n$  by  $g^n e^{im\theta}$ , and get

$$\begin{aligned} & \frac{g^{n+1} e^{im\theta} - g^n \frac{1}{2} [e^{i(m+1)\theta} + e^{i(m-1)\theta}]}{k} + a \frac{g^n e^{i(m+1)\theta} - g^n e^{i(m-1)\theta}}{2h} \\ &= g^n e^{im\theta} \left[ \frac{g - \frac{1}{2} [e^{i\theta} + e^{-i\theta}]}{k} + a \frac{e^{i\theta} - e^{-i\theta}}{2h} \right] = 0 \end{aligned}$$

Now,  $e^{i\theta} - e^{-i\theta} = 2i \sin \theta$ , and  $e^{i\theta} + e^{-i\theta} = 2 \cos \theta$ , so

$$g(\theta) = \cos \theta - ia\lambda \sin \theta, \quad |g(\theta)|^2 = \cos^2 \theta + (a\lambda)^2 \sin^2 \theta$$

Hence, this scheme is **stable**, as long as  $|a\lambda| \leq 1$ .

## Example: Lax-Friedrichs Scheme... Again

The Lax-Friedrichs scheme applied to the equation

$$u_t + au_x - \mathbf{u} = 0,$$

i.e.

$$\frac{v_m^{n+1} - \frac{1}{2} [v_{m+1}^n + v_{m-1}^n]}{k} + a \frac{v_{m+1}^n - v_{m-1}^n}{2h} - \mathbf{v}_m^n = 0,$$

gives rise to the amplification factor

$$g(\theta, k, h) = \cos \theta - ia\lambda \sin \theta + \mathbf{k},$$

with

$$|g(\theta, k, h)|^2 = (\cos \theta + \mathbf{k})^2 + (a\lambda)^2 \sin^2 \theta.$$

For which  $|g(\theta, k, h)|^2 \leq (1 + k)^2 = 1 + 2k + \mathcal{O}(k^2)$  if  $|a\lambda| \leq 1$ .

This scheme is **stable** according to the first inequality in the theorem.

## Modified Schemes and Stability

### Corollary (Stability for Modified Schemes)

*If a scheme as in the **von Neumann stability theorem** is modified so that the modifications result only in the addition to the amplification factor of terms that are  $\mathcal{O}(k)$  uniformly in  $\xi$ , then the modified scheme is stable **if and only if** the original scheme is stable.*

**Proof:** If  $g$  is the amplification factor for the scheme and satisfies  $|g| \leq 1 + Kk$ , then the amplification factor of the modified scheme,  $g'$ , satisfies

$$|g'| = |g + \mathcal{O}(k)| \leq 1 + Kk + Ck = 1 + K'k.$$

Hence the modified scheme is stable **if and only if** the original scheme is stable, and vice versa.  $\square$

## Stability For the One-Way Wave Equation with a Lower-Order Term

### Theorem

*A consistent one-step scheme for the equation*

$$u_t + au_x + bu = 0$$

*is stable if and only if it is stable for this equation when  $b = 0$ . Moreover, when  $k = \lambda h$ , and  $\lambda$  is a constant, the stability condition on  $g(h\xi, k, h)$  is*

$$|g(\theta, 0, 0)| \leq 1.$$

Because of this theorem, it is usually sufficient to consider  $g(h\xi, k, h) \rightsquigarrow g(\theta)$ , and ignore the dependence on  $h$ , and  $k$ .

## Informal Homework

Not Due

Study the examples in chapter 2, and the proofs of the theorems.

Read § 2.3 — **Comments on Instability and Stability.**