Numerical Solutions to PDEs

Lecture Notes #4
— Analysis of Finite Difference Schemes —
Fourier Analysis; Von Neumann Analysis

Peter Blomgren,
⟨`blomgren.peter@gmail.com`⟩

Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

http://terminus.sdsu.edu/

Spring 2018

Previously...

Convergence: The desired result; as we refine the grid, the numerical solution of the Finite Difference Scheme (FDS) should better and better represent the exact (continuous) solution of the PDE.

Consistency: Easily checked by **Taylor expansion** — the expansion of the FDS should give the PDE + terms that go to zero as $(h, k) \to 0$.

Stability: An ℓ_2-energy bound on the solution of the FDS in terms of the initial condition (+ further levels of initialization for multi-step schemes). Hard to check using the definitions — we start developing tools today!

Well-Posedness: A property of the PDE for IVPs — An L^2-energy bound on the solution in terms of the initial conditions.

The CFL-condition

Courant-Friedrichs-Lewy condition $|a\lambda| \leq 1$ (for explicit one-step schemes applied to $u_t + au_x + bu = f$) is **necessary** (but not sufficient) for stability. It expresses the need for the numerical speed of propagation λ^{-1} to match or exceed the physical speed of propagation a.

Theorem (The Lax-Richtmyer Equivalence Theorem)

A consistent finite difference scheme for a partial differential equation for which the initial value problem is well-posed is convergent if and only if it is stable.

Fourier Analysis: An Applied Crash Course
The Road to Stability
Stability

Fourier transform & Fourier inversion formula
Fourier Transform on a Grid
Parseval's Relations

A Crash Course in Fourier Analysis
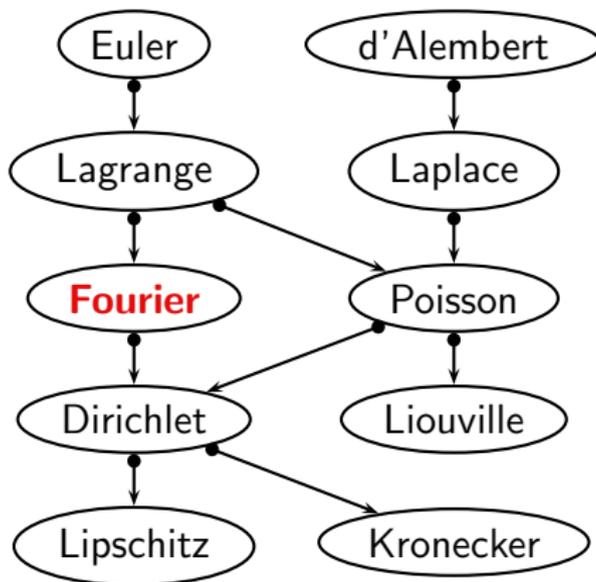

Figure: Jean-Baptiste Joseph Fourier (21 March 1768 – 16 May 1830). Advisor: Joseph Louis Lagrange. Student: Gustav Peter Lejeune Dirichlet ($+1$).

Fourier Analysis: An Applied Crash Course
The Road to Stability
Stability

Fourier transform & Fourier inversion formula
Fourier Transform on a Grid
Parseval's Relations

Truncated Genealogy   (Advisor → Student)

Fourier Analysis: An Applied Crash Course
The Road to Stability
Stability

Fourier transform & Fourier inversion formula
Fourier Transform on a Grid
Parseval's Relations

A Crash Course in Fourier Analysis

The **Fourier transform** of a function $u(x)$, $x \in \mathbb{R}$ is defined by

$$\widehat{u}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} u(x) \, dx.$$

The **Fourier inversion formula**

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \widehat{u}(\omega) \, d\omega,$$

recovers the function from its Fourier transform.

Essentially, the Fourier transform representation expresses $u(x)$ as an infinite superposition of (complex) waves
$e^{i\omega x} = \cos(\omega x) + i \sin(\omega x)$, with amplitudes $\widehat{u}(\omega)$.

(!) $u(x)$ and $\widehat{u}(\omega)$ must satisfy certain criteria for the integrals (above) to be well-defined. We sweep those details under the rug, and refer to **Math 668: Applied Fourier Analysis**.

Fourier Analysis: An Applied Crash Course
The Road to Stability
Stability

Fourier transform & Fourier inversion formula
Fourier Transform on a Grid
Parseval's Relations

Example   With Correction of Typo (Strikwerda p.38)

We consider the function

$$u(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0. \end{cases}$$

The Fourier transform of $u(x)$ is given by

$$\underbrace{\widehat{u}(\omega)}_{\text{correction}} = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-i\omega x} e^{-x} \, dx = \frac{1}{\sqrt{2\pi}} \frac{1}{1 + i\omega}.$$

The tools needed for evaluation of such integrals can be found in
Math $532 + 631A + 631B$: Complex Analysis.

Tables of Fourier transforms can be found online in various places;
ask uncle Google for guidance.

Fourier Analysis: An Applied Crash Course
The Road to Stability
Stability

Fourier transform & Fourier inversion formula
Fourier Transform on a Grid
Parseval's Relations

Fourier Transform Tables: A Warning

There are several ways of defining the Fourier transform — the normalization constants for the forward and inverse transforms are chosen from one of the following three set

$$\left\{ \frac{1}{\sqrt{2\pi}},\ \frac{1}{\sqrt{2\pi}} \right\}, \quad \left\{ 1,\ \frac{1}{2\pi} \right\}, \quad \left\{ \frac{1}{2\pi},\ 1 \right\},$$

and the factors in the integrals can be chosen to be

$$\left\{ e^{-i\omega x},\ e^{i\omega x} \right\}, \quad \left\{ e^{i\omega x},\ e^{-i\omega x} \right\}.$$

For a total of six "natural" ways to define the transform and its inverse. Of course, mathematicians and engineers have agreed to disagree on the definition of the One True Fourier Transform™. — These choices also affect numerical implementations of the discrete Fourier transform...

Fourier Analysis: An Applied Crash Course
The Road to Stability
Stability

Fourier transform & Fourier inversion formula
Fourier Transform on a Grid
Parseval's Relations

Extending the Fourier Transform to Grid Functions, I

For a grid function v_m defined for all integers coordinates m, the Fourier transform is given by

$$\widehat{v}(\xi) = \frac{1}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-im\xi} \, v_m,$$

where $\xi \in [-\pi, \pi]$, and $\widehat{v}(\pi) = \widehat{v}(-\pi)$.

The inversion formula is given by

$$v_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{im\xi} \, \widehat{v}(\xi) \, d\xi.$$

Fourier Analysis: An Applied Crash Course
The Road to Stability
Stability

Fourier transform & Fourier inversion formula
Fourier Transform on a Grid
Parseval's Relations

Extending the Fourier Transform to Grid Functions, II

For a grid function v_m defined for all coordinates $x_m = h \cdot m$, the Fourier transform is given by

$$\widehat{v}(\xi) = \frac{h}{\sqrt{2\pi}} \sum_{m=-\infty}^{\infty} e^{-imh\xi} \, v_m$$

where $\xi \in [-\pi/h, \pi/h]$, and $\widehat{v}(\pi/h) = \widehat{v}(-\pi/h)$.

The inversion formula is given by

$$v_m = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi} \, \widehat{v}(\xi) \, d\xi.$$

Fourier Analysis: An Applied Crash Course
The Road to Stability
Stability

Fourier transform & Fourier inversion formula
Fourier Transform on a Grid
Parseval's Relations

Parseval's Relations: Preservation of L^2 Energy

With the following definition for the L^2 (continuous) energy

$$\|u\|_2 = \sqrt{\int_{-\infty}^{\infty} |u(x)|^2 \, dx},$$

the following holds

Parseval's Relations

$$\sqrt{\int_{-\infty}^{\infty} |u(x)|^2 \, dx} = \|u\|_2 = \|\widehat{u}\|_2 = \sqrt{\int_{-\infty}^{\infty} |\widehat{u}(\omega)|^2 \, d\omega}$$

$$\sqrt{h \sum_{m=-\infty}^{\infty} |v_m|^2} = \|v\|_2 = \|\widehat{v}\|_2 = \sqrt{\int_{-\pi/h}^{\pi/h} |\widehat{v}(\xi)|^2 \, d\xi}$$

These relations are key to our stability analysis, and are also a big reason why measuring quantities in the L^2 (and/or ℓ_2) norm is usually a Good Thing$^{\text{TM}}$ — many times the norm expresses a natural physical energy, *and* that energy is preserved under the Fourier transform.

Fourier Analysis: An Applied Crash Course
The Road to Stability
Stability

Using Parseval's Relations
Von Neumann Analysis
Von Neumann Stability

The Road to Stability: Using Parseval's Relations

Using Parseval's relations, we can rewrite the inequalities that appeared in the definition of stability (last lecture)

$$h \sum_{m=-\infty}^{\infty} |v_m^n|^2 \leq C_T h \sum_{j=0}^{J} \sum_{m=-\infty}^{\infty} \left| v_m^j \right|^2,$$

and

$$\|v^n\|_h \leq \left[C_T \sum_{j=0}^{J} \|v^j\|_h^2 \right]^{1/2} \quad \Leftrightarrow \quad \|v^n\|_h \leq C_T^* \sum_{j=0}^{J} \|v^j\|_h,$$

by the equivalent inequality (applied in the Fourier domain...)

$$\|\widehat{v}^n\|_h \leq C_T^* \sum_{j=0}^{J} \|\widehat{v}^j\|_h.$$

So???

Fourier Analysis: An Applied Crash Course
The Road to Stability
Stability

Using Parseval's Relations
Von Neumann Analysis
Von Neumann Stability

Fourier Analysis and PDEs

Given the Fourier inversion formula

$$u(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \, \widehat{u}(\omega) \, d\omega,$$

we formally compute the derivative with respect to x:

$$\frac{\partial u(x)}{\partial x} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \, i\omega \widehat{u}(\omega) \, d\omega.$$

This leads us to the stunningly simple, and extremely useful conclusion that

$$\widehat{\left(\frac{\partial u}{\partial \mathbf{x}} \right)} = \mathbf{i}\omega \, \widehat{u}(\omega)$$

i.e. **differentiation** on the physical side, corresponds to **multiplication by** $\mathbf{i}\omega$ on the Fourier transform side.

Fourier Analysis: An Applied Crash Course
The Road to Stability
Stability

Using Parseval's Relations
Von Neumann Analysis
Von Neumann Stability

Higher Derivatives — L^2 — Parseval

It now follows that the squared L^2-norm of the r-th derivative is given by

$$\int_{-\infty}^{\infty} \left| \frac{\partial^r u(x)}{\partial x^r} \right|^2 dx = \int_{-\infty}^{\infty} |\omega|^{2r} |\widehat{u}(\omega)|^2 d\omega,$$

These quantities exist (*i.e.* u has L^2 integrable derivatives of order through r, if and only if

$$\int_{-\infty}^{\infty} (1 + |\omega|^2)^r |\widehat{u}(\omega)|^2 d\omega < \infty.$$

From this we can define the **function space** (**Sobolev space**, also denoted $W_2^r(\mathbb{R})$, or $W^{r,2}(\mathbb{R})$) $H^r(\mathbb{R})$ ($r > 0$) as the set of functions $u \in L^2(\mathbb{R})$, for which (note $H^0(\mathbb{R}) \equiv L^2(\mathbb{R})$)

$$\|u\|_{H^r} = \sqrt{\int_{-\infty}^{\infty} (1 + |\omega|^2)^r |\widehat{u}(\omega)|^2 d\omega} < \infty.$$

Fourier Analysis: An Applied Crash Course
The Road to Stability
Stability

Using Parseval's Relations
Von Neumann Analysis
Von Neumann Stability

Notations for Norms of Derivatives

We introduce the notation

$$\|D^r u\|^2 = \int_{-\infty}^{\infty} \left| \frac{\partial^r}{\partial x^r} u(x) \right|^2 \, dx = \int_{-\infty}^{\infty} |\omega|^{2r} \, |\widehat{u}(\omega)|^2 \, d\omega,$$

and note (for future reference), that the integral over x is only defined when r is an integer, but the integral over ω can be used for "fractional derivatives."

OK, lets return to the one-way wave equation...

Fourier Analysis: An Applied Crash Course
The Road to Stability
Stability

Using Parseval's Relations
Von Neumann Analysis
Von Neumann Stability

Fourier Analysis and the One-Way Wave Equation, I

Consider, with $u(0, x) = u_0(x)$ specified,

$$u_t + au_x = 0, \quad \Leftrightarrow \quad u_t = -au_x.$$

Fourier transforming in the x-coordinate, we get

$$\widehat{u}_t = -ia\omega\widehat{u}, \quad \widehat{u}_0(\omega) \text{ given.}$$

This is an Ordinary Differential Equation (ODE) in t, and the solution is given by

$$\widehat{u}(t, \omega) = e^{-ia\omega t}\,\widehat{u}_0(\omega).$$

With the help of the tools we have developed, we can show that this Initial Value Problem is well-posed.

Fourier Analysis: An Applied Crash Course
The Road to Stability
Stability

Using Parseval's Relations
Von Neumann Analysis
Von Neumann Stability

Fourier Analysis and the One-Way Wave Equation, II

We have, using Parseval's equality

$$
\int_{-\infty}^{\infty} |u(t,x)|^2 \, dx = \int_{-\infty}^{\infty} |\widehat{u}(t,\omega)|^2 \, d\omega = \int_{-\infty}^{\infty} |e^{-ia\omega t} \, \widehat{u_0}|^2 \, d\omega =
$$

$$
\int_{-\infty}^{\infty} \underbrace{|e^{-ia\omega t}|^2}_{1} |\widehat{u_0}|^2 \, d\omega = \int_{-\infty}^{\infty} |\widehat{u_0}|^2 \, d\omega = \int_{-\infty}^{\infty} |u_0|^2 \, dx = \|u_0\|_2^2.
$$

Hence, not only do we have a bound on the energy — we have an exact value, which does not change in time. $\Rightarrow$ **The IVP is well-posed**.

Fourier Analysis: An Applied Crash Course
The Road to Stability
Stability

Using Parseval's Relations
Von Neumann Analysis
Von Neumann Stability

Von Neumann Analysis

The application of Fourier analysis which presently is of interest to us is the application to the stability analysis of finite difference schemes; known as **von Neumann analysis**.

Starting from the forward-time-backward-space scheme (suitable only when $a > 0$, think about the characteristic) applied to the one-way wave equation ($u_t + au_x = 0$):

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_m^n - v_{m-1}^n}{h} = 0.$$

We rewrite this in the form ($\lambda = k/h$)

$$v_m^{n+1} = (1 - a\lambda)v_m^n + a\lambda v_{m-1}^n.$$

Next we, use the Fourier inversion formula to represent the quantities on the right-hand side....

Fourier Analysis: An Applied Crash Course
The Road to Stability
Stability

Using Parseval's Relations
Von Neumann Analysis
Von Neumann Stability

Von Neumann Analysis... Moving Along   1 of 2

With

$$v_m^n = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi} \, \widehat{v}^n(\xi) \, d\xi,$$

we get

$$v_m^{n+1} = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi} \left[(1 - a\lambda) + a\lambda \underbrace{e^{-ih\xi}}_{\text{from } v_{m-1}^n} \right] \widehat{v}^n(\xi) \, d\xi.$$

From the inversion formula we also have

$$v_m^{n+1} = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{\pi/h} e^{imh\xi} \, \widehat{v}^{n+1}(\xi) \, d\xi.$$

We have two representations of the same quantity...

Fourier Analysis: An Applied Crash Course
The Road to Stability
Stability

Using Parseval's Relations
Von Neumann Analysis
Von Neumann Stability

Von Neumann Analysis... Moving Along   2 of 2

The integrands must be the same, hence

$$\widehat{v}^{n+1}(\xi) = \underbrace{\left[(1 - a\lambda) + a\lambda\, e^{-ih\xi}\right]}_{g(h\xi)} \widehat{v}^n(\xi).$$

$g(h\xi)$ is known as the **amplification factor**, and we note that

$$\widehat{v}^n(\xi) = g(h\xi)^n\, \widehat{v}^0(\xi).$$

If $|g(h\xi)| > 1$, then the energy grows exponentially; hence **for stability we must require** $|g(h\xi)| \leq 1$.

Fourier Analysis: An Applied Crash Course
The Road to Stability
Stability

Using Parseval's Relations
Von Neumann Analysis
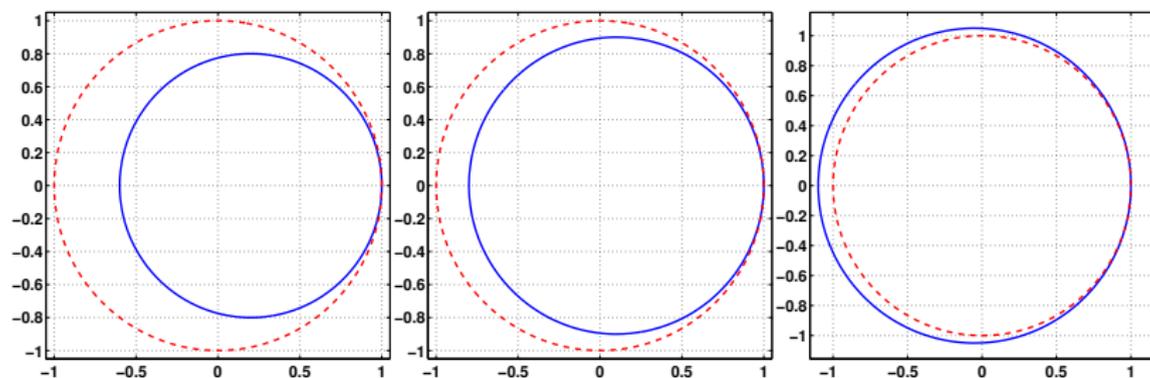Von Neumann Stability

Von Neumann Analysis... Closing Out

We let $\theta = h\xi$, and use $e^{-i\theta} = \cos\theta - i\sin\theta$, and consider $|g(\theta)|^2$:

$$
\begin{aligned}
|g(\theta)|^2 &= (1 - a\lambda + a\lambda\cos\theta)^2 + a^2\lambda^2\sin^2\theta \\
&= \left(1 - 2a\lambda\sin^2\left(\frac{\theta}{2}\right)\right)^2 + 4a^2\lambda^2\sin^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right) \\
&= 1 - 4a\lambda\sin^2\left(\frac{\theta}{2}\right) + 4a^2\lambda^2\sin^4\left(\frac{\theta}{2}\right) + 4a^2\lambda^2\sin^2\left(\frac{\theta}{2}\right)\cos^2\left(\frac{\theta}{2}\right) \\
&= 1 - 4a\lambda(1 - a\lambda)\sin^2\left(\frac{\theta}{2}\right).
\end{aligned}
$$

Since $\sin^2\left(\frac{\theta}{2}\right) \geq 0$, we must require $a\lambda \geq 0$ and $a\lambda \leq 1$ in order for $|g(\theta)|^2 \leq 1$. Hence, the scheme is stable for $\mathbf{0 \leq a\lambda \leq 1}$.

$$
1 - \cos\theta = 2\sin^2\left(\frac{\theta}{2}\right), \quad \sin\theta = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)
$$

Fourier Analysis: An Applied Crash Course
The Road to Stability
Stability

Using Parseval's Relations
Von Neumann Analysis
Von Neumann Stability

Von Neumann Analysis: Images of $g(\theta)$

Figure: Images of $g(\theta)$ in the complex plane. For $a\lambda = 0.8$ and $a\lambda = 0.9$, the image (**blue**, solid) is contained in the unit circle (**red**, dashed), but for $a\lambda = 1.05$ we can clearly see that $|g(\theta)| > 1$, except for $g(0) = 1$.

With this example in mind, we are ready for the formal criterion for stability.

Fourier Analysis: An Applied Crash Course
The Road to Stability
Stability

Using Parseval's Relations
Von Neumann Analysis
Von Neumann Stability

Von Neumann Analysis: The Stability Condition

Theorem (Von Neumann Stability)

A one-step finite difference scheme (with constant coefficients) is stable in a stability region Λ *if and only if there is a constant K (independent of* θ, *k, and h) such that*

$$|g(\theta, k, h)| \leq 1 + Kk$$

with $(k, h) \in \Lambda$. *If* $g(\theta, k, h)$ *is independent of h and k, the stability condition can be replaced with the restricted stability condition*

$$|g(\theta)| \leq 1.$$

Determining stability this way is quite straightforward — only symbolic manipulations of the expression for $|g(\theta, k, h)|^2$ are needed.

Fourier Analysis: An Applied Crash Course
The Road to Stability
Stability

Examples: FTCS, Lax-Friedrichs
Stability of Modified Schemes
Impact of Lower-Order Terms

Example: Forward-Time-Central-Space

The procedure can be stream-lined quite a bit, consider

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0.$$

Replace v_m^n by $g^n e^{im\theta}$, and get

$$\frac{g^{n+1}e^{im\theta} - g^n e^{im\theta}}{k} + a\frac{g^n e^{i(m+1)\theta} - g^n e^{i(m-1)\theta}}{2h}$$

$$= g^n e^{im\theta}\left[\frac{g-1}{k} + a\frac{e^{i\theta} - e^{-i\theta}}{2h}\right] = 0.$$

The expression in the square bracket must be zero, and
$e^{i\theta} - e^{-i\theta} = 2i\sin\theta$, so the amplification factor is given by

$$g(\theta) = 1 - ia\lambda\sin\theta, \quad |g(\theta)|^2 = 1 + (a\lambda)^2\sin^2\theta \geq 1.$$

Hence, this scheme is **unstable**.

Fourier Analysis: An Applied Crash Course
The Road to Stability
Stability

Examples: FTCS, Lax-Friedrichs
Stability of Modified Schemes
Impact of Lower-Order Terms

Example: Lax-Friedrichs Scheme

The Lax-Friedrichs Scheme is quite similar to FT-CS:

$$\frac{v_m^{n+1} - \frac{1}{2}\left[\mathbf{v_{m+1}^n} + \mathbf{v_{m-1}^n}\right]}{k} + a\,\frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0$$

Replace v_m^n by $g^n\,e^{im\theta}$, and get

$$\frac{g^{n+1}e^{im\theta} - g^n\frac{1}{2}\left[\mathbf{e^{i(m+1)\theta}} + \mathbf{e^{i(m-1)\theta}}\right]}{k} + a\,\frac{g^n e^{i(m+1)\theta} - g^n e^{i(m-1)\theta}}{2h}$$

$$= g^n e^{im\theta}\left[\frac{g - \frac{1}{2}\left[\mathbf{e^{i\theta}} + \mathbf{e^{-i\theta}}\right]}{k} + a\,\frac{e^{i\theta} - e^{-i\theta}}{2h}\right] = 0$$

Now, $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$, and $e^{i\theta} + e^{-i\theta} = 2\cos\theta$, so

$$g(\theta) = \cos\theta - ia\lambda\sin\theta, \quad |g(\theta)|^2 = \cos^2\theta + (a\lambda)^2\sin^2\theta$$

Hence, this scheme is **stable**, as long as $|a\lambda| \leq 1$.

Fourier Analysis: An Applied Crash Course
The Road to Stability
Stability

Examples: FTCS, Lax-Friedrichs
Stability of Modified Schemes
Impact of Lower-Order Terms

Example: Lax-Friedrichs Scheme... Again

The Lax-Friedrichs scheme applied to the equation

$$u_t + au_x - \mathbf{u} = 0,$$

i.e.

$$\frac{v_m^{n+1} - \frac{1}{2}\left[v_{m+1}^n + v_{m-1}^n\right]}{k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} - \mathbf{v_m^n} = 0,$$

gives rise to the amplification factor

$$g(\theta, k, h) = \cos\theta - ia\lambda\sin\theta + \mathbf{k},$$

with

$$|g(\theta, k, h)|^2 = (\cos\theta + \mathbf{k})^2 + (a\lambda)^2\sin^2\theta.$$

For which $|g(\theta, k, h)|^2 \leq (1+k)^2 = 1 + 2k + \mathcal{O}\left(k^2\right)$ if $|a\lambda| \leq 1$. This scheme is **stable** according to the first inequality in the theorem.

Fourier Analysis: An Applied Crash Course
The Road to Stability
Stability

Examples: FTCS, Lax-Friedrichs
Stability of Modified Schemes
Impact of Lower-Order Terms

Modified Schemes and Stability

Corollary (Stability for Modified Schemes)

If a scheme as in the **von Neumann stability theorem** *is modified so that the modifications result only in the addition to the amplification factor of terms that are $\mathcal{O}(k)$ uniformly in ξ, then the modified scheme is stable* if and only if *the original scheme is stable.*

Proof: If g is the amplification factor for the scheme and satisfies $|g| \leq 1 + Kk$, then the amplification factor of the modified scheme, g', satisfies

$$|g'| = |g + \mathcal{O}(k)| \leq 1 + Kk + Ck = 1 + K'k.$$

Hence the modified scheme is stable if and only if the original scheme is stable, and vice versa. $\square$

Fourier Analysis: An Applied Crash Course
The Road to Stability
Stability

Examples: FTCS, Lax-Friedrichs
Stability of Modified Schemes
Impact of Lower-Order Terms

Stability For the One-Way Wave Equation with a Lower-Order Term

Theorem

A consistent one-step scheme for the equation

$$u_t + a u_x + b u = 0$$

is stable if and only if it is stable for this equation when $b = 0$. Moreover, when $k = \lambda h$, and λ is a constant, the stability condition on $g(h\xi, k, h)$ is

$$|g(\theta, 0, 0)| \leq 1.$$

Because of this theorem, it is usually sufficient to consider $g(h\xi, k, h) \rightsquigarrow g(\theta)$, and ignore the dependence on h, and k.

Fourier Analysis: An Applied Crash Course
The Road to Stability
Stability

Examples: FTCS, Lax-Friedrichs
Stability of Modified Schemes
Impact of Lower-Order Terms

Informal Homework

Not Due

Study the examples in chapter 2, and the proofs of the theorems.

Read § 2.3 — **Comments on Instability and Stability**.