

# Numerical Solutions to PDEs

## Lecture Notes #7

— Stability for Multistep Schemes —  
Leapfrog Scheme; General Multistep Schemes

Peter Blomgren,  
(blomgren.peter@gmail.com)

Department of Mathematics and Statistics  
Dynamical Systems Group  
Computational Sciences Research Center  
San Diego State University  
San Diego, CA 92182-7720  
<http://terminus.sdsu.edu/>

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## Last Time

We checked the stability of the Lax-Wendroff and Crank-Nicolson schemes, and came up with the following:

	Lax-Wendroff	Crank-Nicolson
Mode	Explicit	Implicit
Order of Accuracy	(2,2)	(2,2)
Stability Criterion	$ a\lambda  \leq 1$ (CFL)	Unconditionally Stable

Difference Notation  $\{\delta_+, \delta_-, \delta_0, \delta^2\}$  and the Difference Calculus, was introduced as a convenient tool to derive higher order schemes.

The main course on the menu was the discussion on **boundary conditions**. For finite difference schemes we must both respect **physical boundary conditions** as well as (sometimes) introduce additional **numerical boundary conditions**. The implementation of these boundary conditions affect both the **order of accuracy**, and **stability** of the scheme.

## Stability for Multistep Schemes: Introduction

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We have seen the necessary and sufficient conditions for the stability of one-step schemes:

Theorem (The CFL Condition)

*For an explicit scheme for the hyperbolic equation*

$$u_t + au_x = 0,$$

*of the form*

$$v_m^{n+1} = \alpha v_{m+1}^n + \beta v_m^n + \gamma v_{m-1}^n,$$

*with  $\lambda = k/h$  held constant, a **necessary condition** for stability is the **Courant-Friedrichs-Lewy (CFL) condition**,*

$$|a\lambda| \leq 1.$$

*For systems of equations for which  $\bar{\mathbf{v}}$  is a vector and  $\alpha$ ,  $\beta$ , and  $\gamma$  are matrices, we must have  $|a_i\lambda| \leq 1$  for all eigenvalues  $a_i$  of the matrix  $A$ .*



## Stability for Multistep Schemes: Introduction

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## Theorem (Von Neumann Stability)

A one-step finite difference scheme (with constant coefficients) is stable in a stability region  $\Lambda$  **if and only if** there is a constant  $K$  (independent of  $\theta$ ,  $k$ , and  $h$ ) such that

$$|g(\theta, k, h)| \leq 1 + Kk,$$

with  $(k, h) \in \Lambda$ . If  $g(\theta, k, h)$  is independent on  $h$  and  $k$ , the stability condition can be replaced with the restricted stability condition

$$|g(\theta)| \leq 1.$$

Now, we **extend this analysis to multi-step schemes**. Starting with the leap-frog scheme, moving to general multi-step schemes. Additional theoretical tools: — the Schur, and von Neumann polynomials which will help us determine stability criteria for multi-step methods.

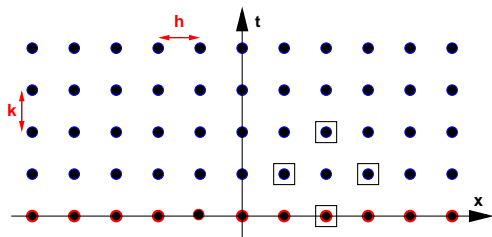


## Stability for the Leapfrog Scheme

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The leapfrog (central-time-central-space) scheme for the homogeneous one-way wave equation is given by

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} + a \frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0.$$



As usual we set  $v_m^n \rightsquigarrow g^n e^{imh\xi}$  (from application of the Fourier inversion formula), and eliminate common factors (here  $g^{n-1} e^{imh\xi}$ ).

## Stability for the Leapfrog Scheme

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We get — ( $h\xi \equiv \theta$ , throughout this lecture) —

$$\begin{aligned}\frac{v_m^{n+1} - v_m^{n-1}}{2k} + a \frac{v_{m+1}^n - v_{m-1}^n}{2h} &= 0 \\ \frac{g^2 - 1}{2k} + a \frac{g(e^{i\theta} - e^{-i\theta})}{2h} &= 0 \\ \mathbf{g^2 - 1 + 2ia\lambda \sin(\theta)g} &= \mathbf{0}\end{aligned}$$

Hence,

$$g_{\pm}(\theta) = -ia\lambda \sin(\theta) \pm \sqrt{1 - (a\lambda)^2 \sin^2(\theta)}. \quad (1)$$

I. When  $\mathbf{g_+ \neq g_-}$ , the solution is given by

$$\hat{v}^n(\xi) = A_+(\xi)g_+(h\xi)^n + A_-(\xi)g_-(h\xi)^n, \quad (2)$$

and  $A_{\pm}(\xi)$  are determined by initial conditions.

## Stability for the Leapfrog Scheme

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Sometimes it is useful to rewrite (2) in the form

$$\widehat{v}^n(\xi) = A(\xi)g_+(h\xi)^n + B(\xi) \left[ \frac{g_-(h\xi)^n - g_+(h\xi)^n}{g_-(h\xi) - g_+(h\xi)} \right], \quad (3)$$

where  $A(\xi)$  and  $B(\xi)$  are determined by initial conditions.

**II.** When  $\mathbf{g}_+ = \mathbf{g}_- = g$ , the solution is given by

$$\widehat{v}^n(\xi) = A(\xi)g(h\xi)^n + n \cdot B(\xi)g(h\xi)^{n-1}, \quad (4)$$

where  $A(\xi)$ , and  $B(\xi)$  are related to  $\widehat{v}^0(\xi)$ , and  $\widehat{v}^1(\xi)$  by

$$\begin{aligned} A(\xi) &= \widehat{v}^0(\xi) \\ B(\xi) &= \widehat{v}^1(\xi) - \widehat{v}^0(\xi)g(h\xi). \end{aligned} \quad (5)$$

We will refer back to these expressions when we analyze the stability of the scheme.



## Stability for the Leapfrog Scheme

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We discuss the stability in terms of

Definition (Stable Scheme)

A finite difference scheme  $P_{k,h}v_m^n = 0$  for a first-order equation is **stable** in a stability region  $\Lambda$  if there is an integer  $J$  such that for any positive time  $T$ , there is a constant  $C_T$  such that

$$h \sum_{m=-\infty}^{\infty} |v_m^n|^2 \leq C_T h \sum_{j=0}^J \sum_{m=-\infty}^{\infty} |v_m^j|^2,$$

for  $0 \leq nk \leq T$ , with  $(k, h) \in \Lambda$ .

with the integer  $J = \mathbf{1}$ .

First, we consider the case where  $g_+ \neq g_-$ , and **choose** the initial conditions so that  $B(\xi) \equiv 0$ .

## Stability for the Leapfrog Scheme

Now, with this setup and using (3) we have

$$|\widehat{v}^n(\xi)| = |A(\xi)| \cdot |g_+(h\xi)|^n,$$

and it follows that we must require

$$|g_+(h\xi)| \leq 1 + Kk,$$

for stability. Application with different initial conditions (such that  $A(\xi) \equiv 0$ ) gives the same restriction on  $g_-(h\xi)$ .

When  $\lambda$  is constant, the restricted conditions

$$|g_{\pm}(h\xi)| \leq 1,$$

apply.

## Stability for the Leapfrog Scheme

From (1), with  $|a\lambda| \leq 1$  we have that

$$|g_{\pm}|^2 = 1 - (a\lambda)^2 \sin^2(\theta) + (a\lambda)^2 \sin^2(\theta) = 1,$$

and when  $|a\lambda| > 1$ , we get

$$|g_-(\pi/2)| = |a\lambda| + \sqrt{(a\lambda)^2 - 1} \geq |a\lambda| > 1.$$

Hence,  $|a\lambda| \leq 1$  is a necessary condition for stability.

But... We're not done. — We must also look at the case  $g_+ = g_-$ . This equality holds only when  $|a\lambda| = 1$ , and  $\theta = \pm\pi/2$ . For these two values we get  $g = \pm i$ , and the solutions

$$\hat{v}^n(\pm\pi/2h) = A(\pm\pi/2h)(\mp i)^n + \mathbf{n} \cdot B(\pm\pi/2h)(\mp i)^{n-1}.$$

Since this term grows linearly in  $n$ , the leapfrog scheme is unstable for  $|a\lambda| = 1$ . **Hence, the leapfrog scheme is stable  $\Leftrightarrow |a\lambda| < 1$ .**

## Initializing the Leapfrog Scheme

The Leapfrog scheme (and other two-step schemes) require that in addition to the initial values  $v_m^0$ , the first time level  $v_m^1$  must also be initialized.

Any consistent one-step scheme, **even an unstable one**, can be used to initialize  $v_m^1$ . Since the unstable scheme is applied only once, the error growth is minimal.

Further, if the grid parameter  $\lambda$  is constant, then the initialization scheme can be **accurate of one order less** than that of the two-step scheme, without degrading the overall accuracy of the scheme.

Thus, we have found a potential use for the unstable forward-time central-space scheme; — as an initializer for the leap-frog scheme.

## Parasitic Modes of the Leapfrog Scheme

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From the expressions

$$\begin{aligned}\widehat{v}^n(\xi) &= A_+(\xi)g_+(h\xi)^n + A_-(\xi)g_-(h\xi)^n, \\ g_{\pm}(\theta) &= -ia\lambda \sin(\theta) \pm \sqrt{1 - (a\lambda)^2 \sin^2(\theta)},\end{aligned}$$

we see that the solution of the leapfrog scheme consists of two parts, associated with  $g_+(\theta)$ , and  $g_-(\theta)$ . We note that  $g_+(0) = 1$ , and  $g_-(0) = -1$ .

We examine how the two parts contribute to the solution.

If we use the forward-time central-space scheme for initialization, then we have

$$\widehat{v}^1(\xi) = (1 - ia\lambda \sin(\theta))\widehat{v}^0(\xi).$$

## Parasitic Modes of the Leapfrog Scheme

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Based on taking the first step using the forward-time central-space scheme, and Taylor expanding the square roots in the expressions for  $g_{\pm}(\theta)$ :

$$\begin{aligned}g_+(\theta) &= 1 - ia\lambda \sin(\theta) - \frac{1}{2}a^2\lambda^2 \sin^2(\theta) + \mathcal{O}(h^4), \\g_-(\theta) &= -1 - ia\lambda \sin(\theta) + \frac{1}{2}a^2\lambda^2 \sin^2(\theta) + \mathcal{O}(h^4),\end{aligned}$$

now, using (5), we get

$$B(\xi) = \left[ \frac{1}{2}a^2\lambda^2 \sin^2(\theta) + \mathcal{O}(\theta^4) \right] \hat{v}^0(\xi).$$

For  $|\theta|$  small,  $|B(\xi)| = \mathcal{O}(\theta^2)$ , *i.e.* small, the scheme behaves like a one-step scheme with amplification factor  $g_+(\theta)$ .



## Parasitic Modes of the Leapfrog Scheme

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When  $|\theta|$  is not small,  $B(\xi)$  is not necessarily small, and the effect of the second amplification factor  $g_-(\theta)$  is felt.

The portion of the solution associated with  $g_-(\theta)$  is called the **parasitic mode**. Since  $g_-(0) = -1$ , the parasitic mode induces rapid **oscillations** in time.

The parasitic mode **travels in the wrong direction**. When  $a$  is positive, the parasitic mode travels to the left, and when  $a$  is negative it travels to the right.

## Example: Parasitic Modes

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We consider the one-way wave-equation, with constant speed  $a = 1$ , in the interval  $[-1, 1]$ , with initial conditions

$$v_m^0 = \begin{cases} \cos^2(\pi x_m) & \text{if } |x_m| \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

At the left boundary ( $x = -1$ ) we set  $v_0^n = 0$  (which is consistent with the equation), and at the right boundary ( $x = 1$ ) we also set  $v_M^0 = 0$  (which is inconsistent with the equation).

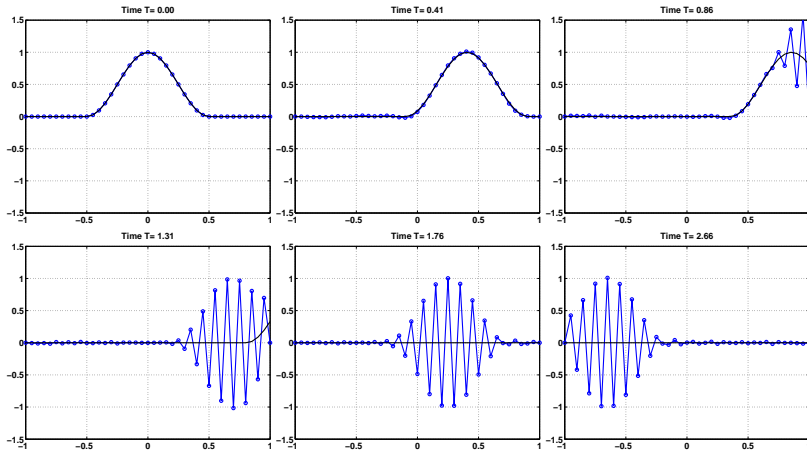
The inconsistent boundary condition will transfer energy into the parasitic mode.

We set the grid parameter  $\lambda = 0.9$ , and  $h = 1/20$ .



## Example: Parasitic Modes

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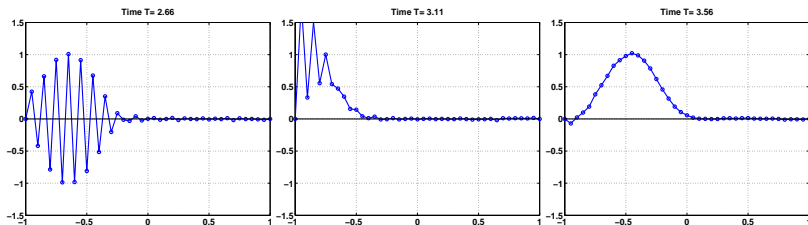


**Figure:** The exact (black solid), and the numerical (blue, with  $\circ$ -markers) solutions. At time  $T=0.86$  (3rd panel), the exact solution is leaving the domain, but the inconsistent boundary condition is starting to pump energy into the parasitic mode, which propagates to the left (panels 4–6).



## Example: Parasitic Modes

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**Figure:** The exact (black solid), and the numerical (blue, with  $\circ$ -markers) solutions. At time  $T=2.66$  (1st panel), the parasitic mode hits the right boundary and bounces back ( $T=3.11$ , 2nd panel), and the reflected energy almost perfectly restores the initial shape of the pulse ( $T=3.56$ , 3rd panel). We note that Dirichlet-type (fixed) boundary conditions are **reflecting** for the wave-equation.

The effects of parasitic modes can be reduced by the use of (numerical) dissipation, which we will discuss next week.

See also **Movie:** [leapfrog\\_ftcs.mpg](#).

## Example: Other Boundary Conditions

We re-run the same problem with different boundary conditions:

Eqn	Boundary Condition	Movie
<b>3.4.1a</b>	$v_M^{n+1} = v_{M-1}^{n+1}$	leapfrog_ftcs_341a.mpg
<b>3.4.1b</b>	$v_M^{n+1} = 2v_{M-1}^{n+1} - v_{M-2}^{n+1}$	leapfrog_ftcs_341b.mpg
<b>3.4.1c</b>	$v_M^{n+1} = v_{M-1}^n$	leapfrog_ftcs_341c.mpg
<b>3.4.1d</b>	$v_M^{n+1} = 2v_{M-1}^n - v_{M-2}^{n-1}$	leapfrog_ftcs_341d.mpg

At first glance (wave leaving the domain) **3.4.1a** and **3.4.1b** seem to perform OK; however, the instability causes the numerical solution to blow up rapidly.

Boundary conditions **3.4.1c** **3.4.1d** are stable, and after the solution leaves the domain only some very minor oscillations remain.

## Stability for General Multistep Schemes

1 of 2

**“The Return of the Symbol”**

We can express the stability of a multistep scheme in several ways, including using the symbol of the scheme:

The stability of a multistep scheme  $P_{k,h}v = R_{k,h}f$  is determined by the roots of the amplification polynomial

$$\Phi(g, \theta) = k p_{k,h} \left( \frac{\ln(g)}{k}, \frac{\theta}{h} \right),$$

or, equivalently

$$\Phi \left( e^{sk}, h\xi \right) = k p_{k,h}(s, \xi).$$

Alternatively, and more familiarly,  $\Phi$  can be obtained by requiring that  $v_m^n = g^n e^{im\theta}$  is a solution to  $P_{k,h}v = 0$ , and  $\Phi(g, \theta)$  is the polynomial of which  $g$  must be a root so that  $v_m^n = g^n e^{im\theta}$  is a solution of  $P_{k,h}v = R_{k,h}f$ .

## Stability for General Multistep Schemes

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We assume that the scheme involves  $\sigma + 1$  time-levels, and therefore  $\Phi$  is a polynomial of degree  $\sigma$ . The integer  $J$  in the stability definition is taken to be  $\sigma$ .

For now, we will largely ignore the relation between  $\Phi$  and the symbol  $p(s, \xi)$ . This relation will, however, be important when we later discuss convergence of multi-step schemes.

OK, our old trick  $v_m^n \rightsquigarrow g^n e^{im\theta}$ , and eliminating common factors will work (pew!).

Still we will run into some trouble.

## A Simple? Example

Consider the multistep scheme for the one-way wave equation

$$\frac{3v_m^{n+1} - 4v_m^n + v_m^{n-1}}{2k} + a \frac{v_{m+1}^{n+1} - v_{m-1}^{n+1}}{2h} = f_m^{n+1}$$

— this scheme is order-(2,2) and unconditionally stable.

The amplification polynomial is

$$\Phi(g, \theta) = \left[ \frac{3 + 2ia\lambda \sin(\theta)}{2} \right] g^2 - 2g + \frac{1}{2}.$$

**Fantastic!** — A second order polynomial with a complex coefficient on the quadratic term; which should be investigated  $\forall \theta$ .

The analysis of this scheme is much harder than that of the leapfrog scheme; we need additional tools from complex analysis and the concepts of Schur, and von Neumann polynomials. This will all be developed in next lecture.

## Moving Along...

## Distinct Roots

Still, we can talk about the stability in general terms: —

If the roots,  $g_\nu$  of  $\Phi(g, \theta)$  are **distinct**, then the solution to the homogeneous difference scheme is given by

$$\hat{v}^n = \sum_{\nu=1}^{\sigma} g_\nu (h\xi)^n A_\nu(\xi), \quad A_\nu(\xi) \text{ determined by initial conditions.}$$

The stability condition is

$$|g_\nu(h\xi)| \leq 1 + Kk, \quad \nu = 1, \dots, \sigma.$$

When  $\Phi(g, \theta)$  is independent of  $k$  and  $h$ , we can set  $K = 0$ .

Roots of Multiplicity  $> 1$ 

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We now look at the case when  $\Phi(g, \theta)$  has **roots of higher multiplicity**. For simplicity, let's assume that  $\Phi(g, \theta)$  is independent of  $k$  and  $h$  so that the restricted stability criterion can be used.

Suppose  $g_1(\theta_0)$  is a multiple root of  $\Phi(g, \theta)$  at  $\theta_0$ ; then

$$\widehat{v}_m^n = [g_1(\theta_0)^n B_0 + n g_1(\theta_0)^{n-1} B_1] e^{im\theta_0},$$

is a solution of the difference equation.

If  $B_0 = 0$  (carefully selected initial conditions), then

$$|\widehat{v}_m^n| = n |g_1(\theta_0)|^{n-1} |B_1|.$$

When  $|g_1(\theta_0)| < 1$ , we have

$$|\widehat{v}_m^n| \leq C \left[ |g_1(\theta_0)| \log \left( \frac{1}{|g_1(\theta_0)|} \right) \right]^{-1} |B_1|.$$



Roots of Multiplicity  $> 1$ 

2 of 2

When  $|\mathbf{g}_1(\theta_0)| = \mathbf{1}$ , we cannot find a bound on  $|\widehat{v}_m^n|$ , and there exists a solution which is linearly unbounded; hence the scheme is unstable in this case.

We have the following

Theorem (Stability of Multistep Schemes)

*If the amplification polynomial  $\Phi(g, \theta)$  is explicitly independent of  $h$  and  $k$ , then the necessary and sufficient condition for the finite difference scheme to be stable is that all roots,  $g_\nu(\theta)$ , satisfy the following conditions:*

- (a)**  $|g_\nu(\theta)| \leq 1$ , and
- (b)** if  $|g_\nu(\theta)| = 1$ , then  $g_\nu(\theta)$  must be a simple root.



## A More General Stability Theorem

### Theorem (Stability of Multistep Schemes)

*A finite difference scheme for a scalar equation is stable **if and only if** all the roots,  $g_\nu(\theta)$ , of the amplification polynomial  $\Phi(g, \theta, k, h)$  satisfy the following conditions:*

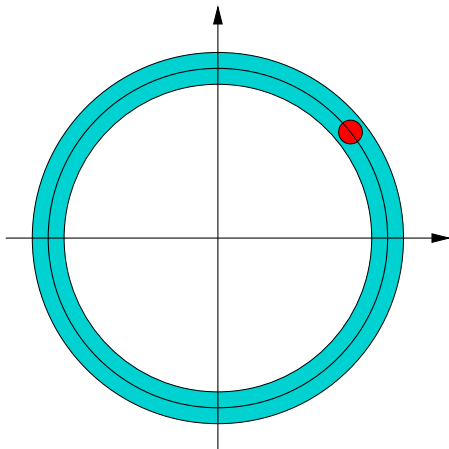
- (a)** *There is a constant  $K$  such that  $|g_\nu| \leq 1 + Kk$ .*
- (b)** *There are positive constants  $c_0$  and  $c_1$  such that if  $c_0 \leq |g_\nu| \leq 1 + Kk$ , then  $g_\nu$  is a simple root, and for any other root  $g_\mu$ , the relation*

$$|g_\nu - g_\mu| \geq c_1$$

*holds for  $h$  and  $k$  sufficiently small.*



## Illustration of the Theorem



**Figure:** In the band  $c_0 \leq r \leq 1 + Kk$ , we can only have simple roots; and the minimal distance between a root in this band and another root is  $c_1$ .

Example: The Leapfrog Scheme

$a\lambda \in \{0.9, 0.99, 1.0, 1.05\}$

