

Numerical Solutions to PDEs

Lecture Notes #8 — Stability for Multistep Schemes — Schur and von Neumann Polynomials

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Recap

In a previous episode of Math 693b...

Previously...

We looked at stability for multistep schemes. — First, we did a complete analysis of the stability picture for the leapfrog scheme,

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} + a \frac{v_{m+1}^n - v_{m-1}^n}{2h} = 0$$

in which we found bounds for the roots of

$$g(\theta)^2 + \left[2ia\lambda \sin(\theta) \right] g(\theta) - 1 = 0$$

so that $|g_{\pm}(\theta)| \leq 1$ for simple roots and $|g_{\pm}(\theta)| < 1$ for multiple roots.

The analysis for general multi-step scheme has the same “flavor,” but we postponed the development of a unified framework for that analysis until today.



Outline

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 - Beyond “Proof by Picture” — Building a Theoretical Toolbox
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Stability

Schur and von Neumann Polynomials

“Proof” by Picture...

Beyond “Proof by Picture” — Building a Theoretical Toolbox

Example: Unnamed Scheme From Last Time

1 of 2

Last time, we boldly stated that the scheme

$$\frac{3v_m^{n+1} - 4v_m^n + v_m^{n-1}}{2k} + a \frac{v_{m+1}^{n+1} - v_{m-1}^{n+1}}{2h} = f_m^{n+1}$$

with amplification polynomial

$$\Phi(g, \theta) = \left[\frac{3 + 2ia\lambda \sin(\theta)}{2} \right] g^2 - 2g + \frac{1}{2}$$

is unconditionally stable, and order-(2,2) accurate.

Whereas pictures are not proof, the plots of the roots for various values of $a\lambda$ and $\theta \in [-\pi, \pi]$ shown on slide 7 seem to indicate that the stability is indeed unconditional.



Example: Unnamed Scheme From Last Time

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Sure, we can take the amplification polynomial

$$\Phi(g, \theta) = \underbrace{\left[\frac{3 + 2ia\lambda \sin(\theta)}{2} \right]}_a g^2 \underbrace{-2}_b g + \underbrace{\frac{1}{2}}_c = 0$$

and formally apply the quadratic formula

$$g_{\pm}(\theta) = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{4 \pm \sqrt{2 - 4 \left[\frac{3 + 2ia\lambda \sin(\theta)}{2} \right] \frac{1}{2}}}{2 \left[\frac{3 + 2ia\lambda \sin(\theta)}{2} \right]}$$

$$\rightsquigarrow g_{\pm}(\theta) = \frac{2 \pm \sqrt{1 - 2ia\lambda \sin(\theta)}}{3 + 2ia\lambda \sin(\theta)}$$



Example: Unnamed Scheme From Last Time

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$$g_{\pm}(\theta) = \frac{2 \pm \sqrt{1 - 2ia\lambda \sin(\theta)}}{3 + 2ia\lambda \sin(\theta)}$$

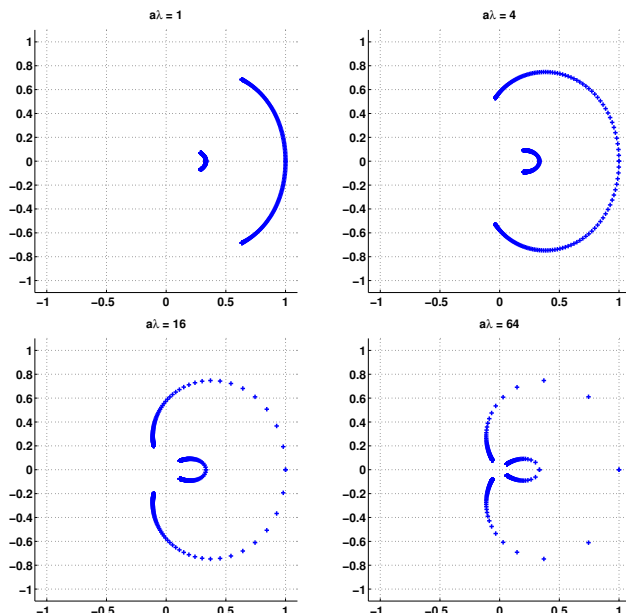
$$\rightsquigarrow g_{\pm}(\theta) = \frac{\left(2 \pm \sqrt{1 - 2ia\lambda \sin(\theta)} \right) (3 - 2ia\lambda \sin(\theta))}{9 + 4(a\lambda)^2 \sin^2(\theta)}$$

Next, define an appropriate branch for the square-root in the complex plane; chase down the various cases... and there it is?!



Example: Unnamed Scheme From Last Time

2 of 2



Example #2: Another Second-Order Accurate Scheme

The second order accurate scheme

$$\frac{7v_m^{n+1} - 8v_m^n + v_m^{n-1}}{6k} + a\delta_0 \left[\frac{2v_m^{n+1} + v_m^n}{3} \right] = f_m^{n+2/3}$$

has the amplification polynomial

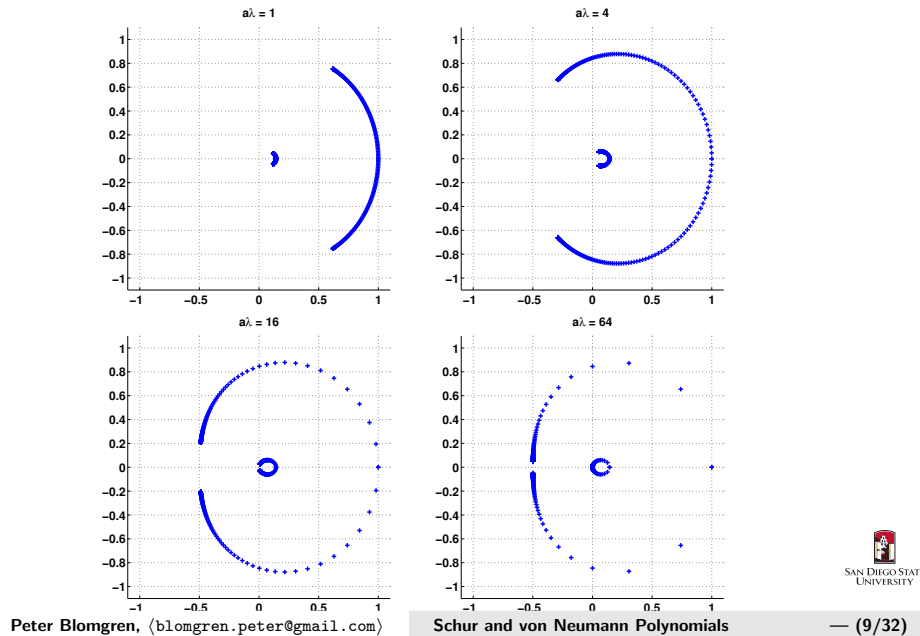
$$\Phi(g) = \left[7 + 4i\beta \right] g^2 - \left[8 - 2i\beta \right] g + 1$$

where $\beta = a\lambda \sin(\theta)$. Also seems to have pretty decent stability properties (see next slide).



Example #2: Root Plots

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Example #3: An Order-(3,4) Accurate Scheme

The (3,4)-order accurate scheme

$$\frac{23v_m^{n+1} - 21v_m^n - 3v_m^{n-1} + v_m^{n-2}}{24k} + \left[1 + \frac{h^2}{6}\delta^2\right]^{-1} \cdot \left[a\delta_0 \left(\frac{v_m^{n+1} + v_m^n}{2} \right) + \frac{k^2 a^2}{8} \delta^2 \left(\frac{v_m^{n+1} - v_m^n}{k} \right) \right] = f_m^{n+1/2},$$

has the amplification polynomial

$$\Phi(g) = \left[23 - 12\alpha + 12i\beta\right]g^3 - \left[21 - 12\alpha - 12i\beta\right]g^2 - 3g + 1,$$

where

$$\alpha = \frac{a^2 \lambda^2 \sin^2(\frac{\theta}{2})}{1 - \frac{2}{3} \sin^2(\frac{\theta}{2})}, \quad \beta = \frac{a \lambda \sin(\theta)}{1 - \frac{2}{3} \sin^2(\frac{\theta}{2})}.$$

Does not seem to be unconditionally stable...

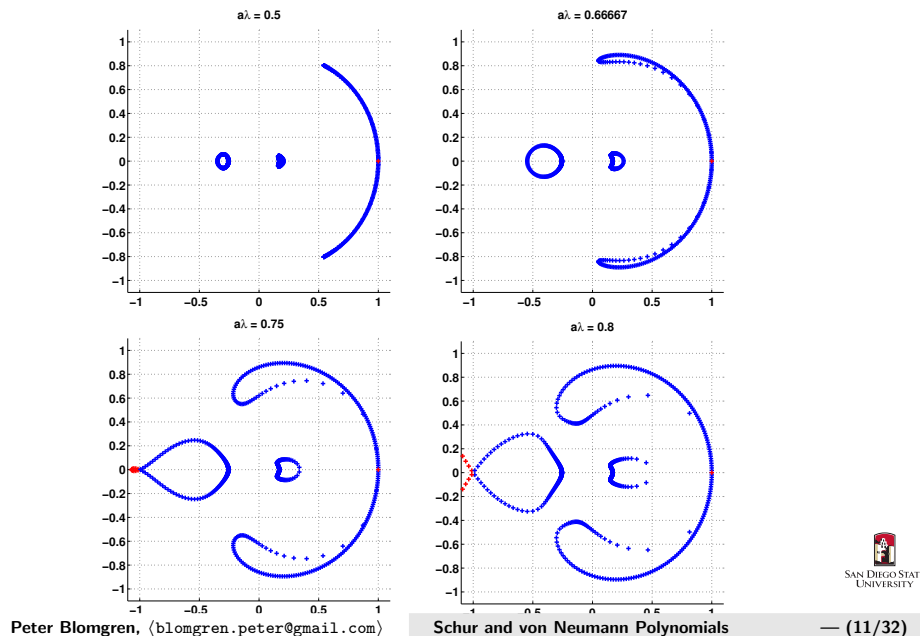
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Schur and von Neumann Polynomials

— (10/32)

Example #3: Root Plots

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Example #4: An Order-(4,4) Accurate Scheme

The (4,4)-order accurate scheme

$$\frac{v_m^{n+2} - v_m^{n-2}}{4k} + a \left[1 + \frac{h^2}{6}\right]^{-1} \delta_0 \left(\frac{2v_m^{n+1} - v_m^n + 2v_m^{n-1}}{3} \right) = \frac{2f_m^{n+1} - f_m^n + 2f_m^{n-1}}{3},$$

has the amplification polynomial

$$\Phi(g) = g^4 + \frac{4}{3}i\beta \left(2g^3 - g^2 + 2g \right) - 1,$$

where

$$\beta = \frac{a \lambda \sin(\theta)}{1 - \frac{2}{3} \sin^2(\frac{\theta}{2})}.$$

Does not seem to be unconditionally stable...

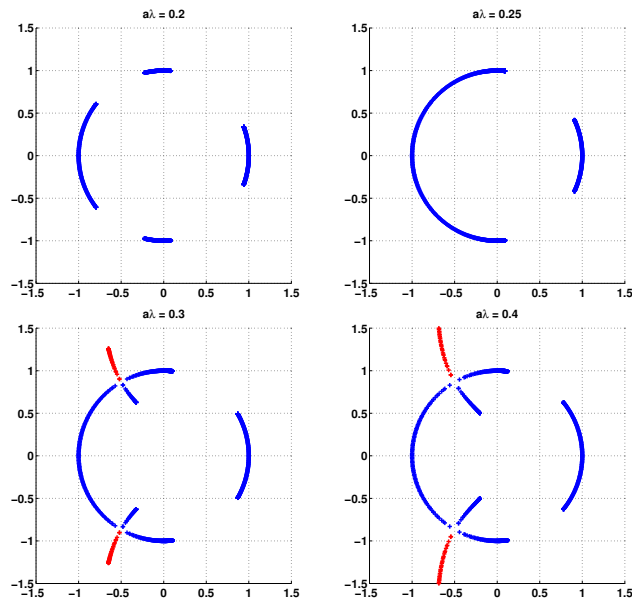
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Example #4: Root Plots

2 of 2



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Schur and von Neumann Polynomials

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Building the Toolbox

Definitions, 1 of 2

Let $\varphi_d(z) = a_d z^d + \dots + a_0 = \sum_{\ell=0}^d a_\ell z^\ell$ be a polynomial of degree d . If $a_d \neq 0$, then φ is of exact degree d .

Definition (Schur Polynomial)

The polynomial φ is a Schur polynomial if all its roots, r_ν , satisfy $|r_\nu| < 1$.

Definition (von Neumann Polynomial)

The polynomial φ is a von Neumann polynomial if all its roots, r_ν , satisfy $|r_\nu| \leq 1$.



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Schur and von Neumann Polynomials

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Moving Beyond "Proof By Picture"

Looking at the expressions and corresponding figures in the previous examples, it is quite clear that the analysis, *i.e.* the determination and bounding of the roots of these polynomials is quite a task.

The good news is that there is a well-developed theory and an algorithm for checking whether the roots of these polynomials satisfy the stability conditions: —

Theorem (Stability of Multistep Schemes)

If the amplification polynomial $\Phi(g, \theta)$ is explicitly independent of h and k , then the necessary and sufficient condition for the finite difference scheme to be stable is that all roots, $g_\nu(\theta)$, satisfy the following conditions:

- (a) $|g_\nu(\theta)| \leq 1$, and
- (b) if $|g_\nu(\theta)| = 1$, then $g_\nu(\theta)$ must be a simple root.

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Schur and von Neumann Polynomials

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Building the Toolbox

Definitions, 2 of 2

Let $\varphi_d(z) = a_d z^d + \dots + a_0 = \sum_{\ell=0}^d a_\ell z^\ell$ be a polynomial of degree d . If $a_d \neq 0$, then φ is of exact degree d .

Definition (Simple von Neumann Polynomial)

The polynomial φ is a simple von Neumann polynomial if φ is a von Neumann polynomial, and its roots on the unit circle are simple roots.

Definition (Conservative Polynomial)

The polynomial φ is a conservative polynomial if all its roots lie on the unit circle, *i.e.* $|r_\nu| = 1$.



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Schur and von Neumann Polynomials

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Building the Toolbox

The Polynomial $\varphi^*(z)$

For a polynomial of exact degree d , we define the polynomial

$$\varphi^*(z) = \sum_{\ell=0}^d \bar{a}_{d-\ell} z^\ell \equiv \overline{\varphi(1/\bar{z})} z^d,$$

where \bar{z} is the complex conjugate of z .

We recursively define the polynomial φ_{d-1} of exact degree $d-1$ by

$$\varphi_{d-1}(z) = \frac{\varphi_d^*(0)\varphi_d(z) - \varphi_d(0)\varphi_d^*(z)}{z} \equiv \frac{\bar{a}_d\varphi_d(z) - a_0\varphi_d^*(z)}{z}.$$

We are now ready to state theorems which provide tests for Schur and simple von Neumann polynomials.



Building the Toolbox

Polynomial Tests

Theorem (Schur Polynomial Test)

φ_d is a Schur polynomial of exact degree d **if and only if** φ_{d-1} is a Schur polynomial of exact degree $d-1$ and $|\varphi_d(0)| < |\varphi_d^*(0)|$.

Theorem (Simple von Neumann Polynomial Test)

φ_d is a simple von Neumann polynomial **if and only if** either

- (a) $|\varphi_d(0)| < |\varphi_d^*(0)|$ and φ_{d-1} is a simple von Neumann polynomial, **or**
- (b) φ_{d-1} is identically zero and φ'_d is a Schur polynomial.

The (somewhat lengthy) proofs, which depend on **Rouché's theorem** (complex analysis) are in Strikwerda pp. 110–114.



Building the Toolbox

3 More Theorems

Theorem (von Neumann Polynomial Test)

φ_d is a von Neumann polynomial of degree d , **if and only if** either

- (a) $|\varphi_d(0)| < |\varphi_d^*(0)|$ and φ_{d-1} is a von Neumann polynomial of degree $d-1$, **or**
- (b) φ_{d-1} is identically zero and φ'_d is a von Neumann polynomial.

Theorem (Conservative Polynomial Test)

φ_d is a conservative polynomial **if and only if** φ_{d-1} is identically zero and φ'_d is a von Neumann polynomial.

Theorem (Simple Conservative Polynomial Test)

φ_d is a simple conservative polynomial **if and only if** φ_{d-1} is identically zero and φ'_d is a Schur polynomial.



Example #1: Revisited

1 of 2

The scheme had the amplification polynomial

$$\varphi_2(z) = \left[\frac{3 + 2ia\lambda \sin(\theta)}{2} \right] z^2 - 2z + \frac{1}{2}.$$

It is stable exactly when $\varphi_2(z)$ is a simple von Neumann polynomial. We make repeated use of the simple-von-Neumann-polynomial-test in order to check stability of the scheme.

We first test $|\varphi_2(\mathbf{0})|^2 = \frac{1}{4} < \frac{1}{4} (3^2 + 4a^2\lambda^2 \sin^2(\theta)) = |\varphi_2^*(\mathbf{0})|^2$, then define, with $(c + di)$ being the coefficient in front of z^2 in $\varphi_2(z)$:

$$\begin{aligned} \varphi_1(z) &= \frac{1}{z} \left[(c - di) \left((c + di)z^2 - 2z + \frac{1}{2} \right) - \frac{1}{2} \left((c - di) - 2z + \frac{1}{2}z^2 \right) \right] \\ &= \left(d^2 + c^2 - \frac{1}{4} \right) z + (1 - 2c + 2id) \end{aligned}$$



Example #1: Revisited

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Now, $\varphi_1(z)$ is a simple von Neumann polynomial as long as

$$\left(d^2 + c^2 - \frac{1}{4}\right)^2 \geq (1 - 2c)^2 + 4d^2 = 1 + 4c^2 + 4d^2 - 4c$$

where $c = \frac{3}{2}$, and $d = a\lambda \sin(\theta)$.

Plugging in we must have

$$a^4 \lambda^4 \sin^4(\theta) + 4a^2 \lambda^2 \sin^2(\theta) + 4 \geq 4a^2 \lambda^2 \sin^2(\theta) + 4$$

Which holds strictly for $\sin(\theta) \neq 0$, and with equality when $\sin(\theta) = 0$.

Conclusion: The scheme is unconditionally stable.



Example #2: Revisited

2 of 2

$$\varphi_1(z) = 4((12 + 4\beta^2)z + ((2\beta^2 - 12) + 12i\beta))$$

is a simple von Neumann polynomial **if and only if**

$$|\varphi_1(0)|^2 = |(2\beta^2 - 12) + 12i\beta|^2 = (12 - 2\beta^2)^2 + 12^2\beta^2 \\ = 144 + 96\beta^2 + 4\beta^4 \leq |\varphi_1^*(0)|^2 = (12 + 4\beta^2)^2 = 144 + 96\beta^2 + 16\beta^4$$

The inequality holds strictly as long as $\beta \neq 0$, in which case we get equality.

Note: Since $\varphi_1(z)$ only has **one** root, it is sufficient to bound that root by " ≤ 1 " in order for $\varphi_1(z)$ to be a simple von Neumann polynomial.

Conclusion: The scheme is unconditionally stable.



Example #2: Revisited

1 of 2

The scheme had the amplification polynomial

$$\varphi_2(z) = [7 + 4i\beta]z^2 - [8 - 2i\beta]z + 1$$

it is stable exactly when $\varphi_2(z)$ is a simple von Neumann polynomial. We make repeated use of the simple-von-Neumann-polynomial-test in order to check stability of the scheme.

With $\beta = a\lambda \sin(\theta)$, we first test $|\varphi_2^*(0)| = |7 - 4i\beta| > 1 = |\varphi_2(0)|$, then define

$$\varphi_1(z) = \frac{1}{z} \left[(7 - 4i\beta) ([7 + 4i\beta]z^2 - [8 - 2i\beta]z + 1) \right. \\ \left. - 1 ([7 - 4i\beta] - [8 + 2i\beta]z + z^2) \right] \\ = 4((12 + 4\beta^2)z + ((2\beta^2 - 12) + 12i\beta)).$$



Example #3: Revisited

1 of 4

In this case the amplification polynomial is given by

$$\varphi_3(z) = [23 - 12\alpha + 12i\beta]z^3 - [21 - 12\alpha - 12i\beta]z^2 - 3z + 1$$

where

$$\alpha = \frac{a^2 \lambda^2 \sin^2(\frac{\theta}{2})}{1 - \frac{2}{3} \sin^2(\frac{\theta}{2})} \in [0, 3a^2 \lambda^2], \quad \beta = \frac{a\lambda \sin(\theta)}{1 - \frac{2}{3} \sin^2(\frac{\theta}{2})} \in [-a\lambda\sqrt{3}, a\lambda\sqrt{3}].$$

The first check $|\varphi_3(0)| < |\varphi_3^*(0)|$ can be expressed as $|\varphi_3^*(0)|^2 - |\varphi_3(0)|^2 > 0$, and we get

$$|\varphi_3^*(0)|^2 - |\varphi_3(0)|^2 = 24(2 - \alpha)(11 - 6\alpha) + 12^2\beta^2$$

we see that we must require $0 \leq \alpha \leq \frac{11}{6}$ for stability.



Example #3: Revisited

2 of 4

The polynomial $\varphi_2(z)$ is (after division by the common factor 24)

$$\varphi_2(z) = \left[(11 - 6\alpha)(2 - \alpha) + 6\beta^2 \right] z^2 - 2 \left[(2 - \alpha)(5 - 3\alpha) - 3\beta^2 - (11 - 6\alpha)i\beta \right] z - (2 - \alpha - 2i\beta),$$

and

$$|\varphi_2^*(0)|^2 - |\varphi_2(0)|^2 = 4(5 - 3\alpha) \left[3(2 - \alpha)^3 + \beta^2(13 - 6\alpha) \right] + 36\beta^4.$$

This now requires that $0 \leq \alpha \leq \frac{5}{3} < \frac{11}{6}$ for stability.



Example #3: Revisited

4 of 4

We now have that

$$\alpha = |a\lambda|^2 \underbrace{\frac{\sin^2(\frac{\theta}{2})}{1 - \frac{2}{3}\sin^2(\frac{\theta}{2})}}_{\in [0,3]} \leq \frac{5}{3}$$

and it follows that the scheme is stable **if and only if**

$$|a\lambda| \leq \frac{\sqrt{5}}{3} \approx 0.7454 \dots$$



Example #3: Revisited

3 of 4

Finally, the polynomial $\varphi_1(z)$ is

$$\begin{aligned} \varphi_1(z) = & \left[120 - 252\alpha + 198\alpha^2 - 69\alpha^3 + 9\alpha^4(18\alpha^2 - 69\alpha + 65)\beta^2 + 9\beta^4 \right] z \\ & + 9\beta^4 + 6(5 - 3\alpha)i\beta^3 + (3\alpha - 5)\beta^2 - \left(18\alpha^3 + 102\alpha^2 + 192\alpha - 120 \right) i\beta \\ & - 9\alpha^4 + 69\alpha^3 - 198\alpha^2 + 252\alpha - 120 \end{aligned}$$

The root-condition $|\varphi_1^*(0)|^2 - |\varphi_1(0)|^2 > 0$ translates to

$$12\beta^4(5 - 3\alpha) \left[6\beta^2 + (11 - 6\alpha)(2 - \alpha) \right] > 0$$

This holds in the range $0 \leq \alpha \leq \frac{5}{3}$; our strictest bound on α .



Example #4: Revisited

1 of 3

$$\varphi_4(z) = z^4 + \frac{4}{3}i\beta \left(2z^3 - z^2 + 2z \right) - 1, \quad \beta = \frac{a\lambda \sin(\theta)}{1 - \frac{2}{3}\sin^2(\frac{\theta}{2})} \in [-a\lambda\sqrt{3}, a\lambda\sqrt{3}].$$

Here, $|\varphi_4(0)| = |\varphi_4^*(0)| = 1$. But $\varphi_3(z) \equiv 0$, hence there is still hope, for $\varphi_4(z)$ being a simple von Neumann polynomial. We must test whether $\psi_3(z) = \frac{3}{4}\varphi_4'(z) = 3z^3 + i\beta(6z^2 - 2z + 2)$ is a **Schur** polynomial.

$$|\psi_3^*(0)| - |\psi_3(0)| = 3 - |2\beta| > 0, \text{ as long as } |\beta| < \frac{3}{2}.$$

We form

$$\psi_2(z) = (9 - 4\beta^2)z^2 + (4\beta^2 + 18i\beta)z - 12\beta^2 - 6i\beta$$

$$|\psi_2^*(0)|^2 - |\psi_2(0)|^2 > 0 \text{ if and only if } (9 - 4\beta^2)^2 > (12\beta^2)^2 + (6\beta)^2, \\ \text{which gives } \beta^2 < \frac{9}{64} [\sqrt{41} - 3] < \frac{9}{4}.$$



Example #4: Revisited

2 of 3

Next, we form

$$\psi_1(z) = (81 - 108\beta^2 - 128\beta^4)z + \left([32\beta^4 + 144\beta^2] - i[264\beta^3 - 162\beta] \right)$$

The one root is inside the unit circle only if

$$(81 - 108\beta^2 - 128\beta^4)^2 - ([32\beta^4 + 144\beta^2]^2 + [264\beta^3 - 162\beta]^2) \geq 0.$$

This expression can be factored as

$$3(9 - 4\beta^2)(3 - 16\beta^2) \underbrace{\left(\beta^2(80\beta^2 - 72) + 81 \right)}_{>0} \geq 0.$$

Hence, $\psi_1(z)$ is a Schur polynomial for

$$\beta^2 < \frac{3}{16} < \frac{9}{64} [\sqrt{41} - 3].$$



Example #4: Revisited

3 of 3

Hence, our final stability condition is

$$|\beta| = \frac{|a\lambda \sin(\theta)|}{1 - \frac{2}{3} \sin^2(\frac{\theta}{2})} < \frac{\sqrt{3}}{4}.$$

The maximum occurs when $\cos(\theta) = -1/2$, and the scheme is stable when $|a\lambda| < \frac{1}{4}$.

Note that even though the scheme is implicit, it is **not** unconditionally stable.



Algorithm for von Neumann / Schur Polynomials

Algorithm

Start with $\varphi_d(z)$ of exact degree d , and set NeumannOrder = 0.

while ($d > 0$) **do**

1. Construct $\varphi_d^*(z)$
2. Define $c_d = |\varphi_d^*(0)|^2 - |\varphi_d(0)|^2$. (*)
3. Construct the polynomial $\psi(z) = \frac{1}{z}(\varphi_d^*(0)\varphi_d(z) - \varphi_d(0)\varphi_d^*(z))$.
- 4.1. If $\psi(z) \equiv 0$, then increase NeumannOrder by 1, and set $\varphi_{d-1}(z) := \varphi_d'(z)$.
- 4.2. Otherwise, if the coefficient of degree $d - 1$ in $\psi(z)$ is 0, then the polynomial is **not** a von Neumann polynomial of any order, **terminate algorithm**.
- 4.3. Otherwise, set $\varphi_{d-1}(z) := \psi(z)$.

end-while (decrease d by 1)

(*) Enforce appropriate conditions on c_d .



Comments on the Algorithm

At the end of the algorithm, if the polynomial has not been rejected by 4.2 —

- The polynomial is a von Neumann polynomial of the resulting order (NeumannOrder) provided that all the parameters c_d satisfy the appropriate inequalities. — These inequalities provide the stability conditions.
- For first-order PDEs, the amplification polynomial must be a von Neumann polynomial of order 1 for the scheme to be stable.
- For second-order PDEs, the amplification polynomial must be a von Neumann polynomial of order 2 for the scheme to be stable.
- Schur polynomials are von Neumann polynomials of order 0.

This analysis can be automated using a symbolic toolbox. — Again, we have reduced something complicated to a deterministic “recipe.”

