Numerical Solutions to PDEs

Lecture Notes #9 — Dissipation and Dispersion

Peter Blomgren,
⟨blomgren.peter@gmail.com⟩

Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

http://terminus.sdsu.edu/

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Outline

1. Recap
   - Stability, and von Neumann Polynomials
   - Hyperbolic PDEs: Consistency, Accuracy, Stability...
2. Dissipation
   - Introduction: Leapfrog & Lax-Wendroff
   - Adding Dissipation
3. Dispersion
   - Introduction
   - Dispersion Relations

Previously...

We developed a framework, and in the end an algorithm, for checking the stability of a general multistep scheme. Such a scheme is stable exactly when its amplification polynomial φ is a **simple von Neumann polynomial:** —

Definition (Simple von Neumann Polynomial)

The polynomial φ is a simple von Neumann polynomial if φ is a von Neumann polynomial, and its roots on the unit circle are simple roots.

The derived algorithm involves comparing the magnitude of the first and last coefficients of the polynomial, and then forming a polynomial of one degree less, and recursively applying the same test to this polynomial.

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Looking Back

Our discussion of finite difference schemes for **hyperbolic PDEs** is almost "complete."

Thanks to the **Lax-Richtmyer equivalence theorem** we only need to consider schemes that are both **consistent** and **stable**.

From the discussion last time, we have a very general framework for checking stability, and we get consistency as a "side-effect" of finding the **order of accuracy** of the scheme.

Usually, finding the order of accuracy comes down to a Taylor expansion; and in the hardest cases we can fall back and use the **symbols** of the scheme and PDE (and congruence to zero for homogeneous equations) in order to "mechanize" the analysis.

For **derivation** of high order accurate schemes, we have a symbolic calculus with the difference operators δ_+, δ_-, δ_0, δ^2, etc...

Dissipation and Dispersion

We now look at two additional topics in the context of hyperbolic equations: — Dissipation and Dispersion.

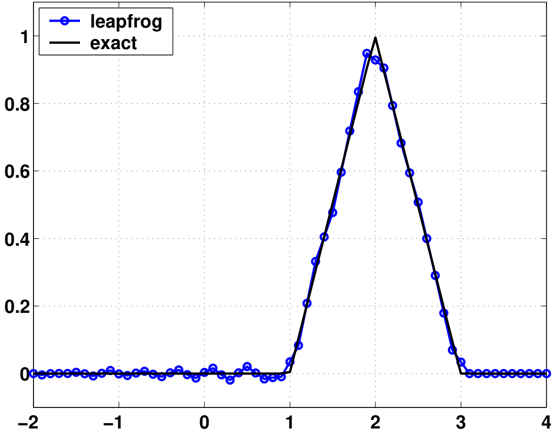
Dissipative schemes: —
Damping out high-frequency waves which make the solution too oscillatory. **Recall** the problem of parasitic modes of the Leapfrog scheme.

Dispersion: — (Numerical Dispersion)
Refers to the fact that finite difference schemes propagate different frequencies at different speeds. — This causes the solution to change shape (spread out) as t grows.

We will return to the issue of *stability of boundary conditions* later in the semester.

Dissipation

The order-(2,2) Leapfrog scheme performs better than the order-(1,2) Lax-Friedrichs scheme, but the solution tends to contain small oscillations.



The Leapfrog Scheme

Consider the leapfrog schemes with oscillatory initial conditions,

$$v_m^j = (-1)^{m+j} \cdot \eta, \quad j = 0, 1, \quad |\eta| \ll 1$$

It is straight-forward to see that the leapfrog solution turns out to be

$$v_m^n = (-1)^{m+n} \cdot \eta$$

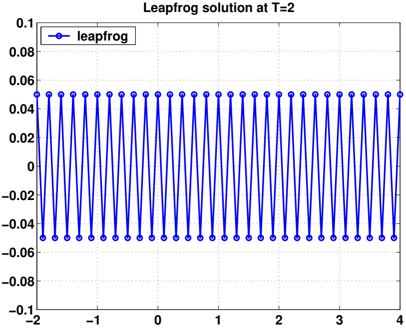


Figure: The leapfrog scheme propagates the oscillatory solution. This means that if we ever, due to numerical error (bad boundary conditions, round-off error, etc) introduce oscillations, they never go away.

The Lax-Wendroff Scheme

Next we consider the Lax-Wendroff scheme with the same oscillatory initial conditions, $v_m^0 = (-1)^m \cdot \eta$

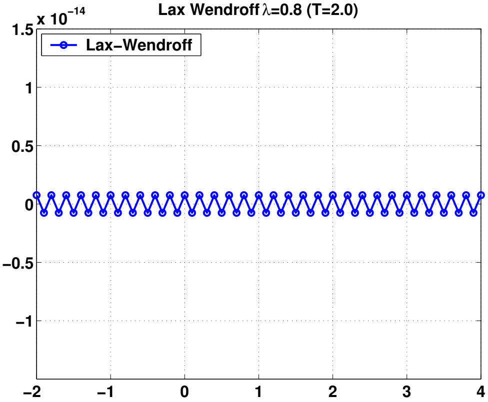


Figure: Here we can see that the magnitude of the high-frequency oscillations have been dampened down to $\sim 10^{-16}$ at time $T = 2$.

The Lax-Wendroff Scheme

The exact solution of the Lax-Wendroff scheme (with this particular initial data) is given by

$$v_m^n = (1 - 2a^2\lambda^2)^n(-1)^{m+n}$$

and since $|a\lambda| < 1$ (here 0.8) we get rapid (exponential) decay of the oscillations. **The Lax-Wendroff scheme is dissipative.**

The Lax-Wendroff Scheme

Definition (Dissipative Scheme)

A scheme is dissipative of order $2r$ if there exists a positive constant c, independent of h and k, such that each amplification factor $g_\nu(\theta)$ satisfies

$$|g_\nu(\theta)| \leq 1 - c \sin^{2r}\left(\frac{\theta}{2}\right) \quad \Leftrightarrow \quad |g_\nu(\theta)|^2 \leq 1 - c^* \sin^{2r}\left(\frac{\theta}{2}\right)$$

The amplification polynomial for the Lax-Wendroff scheme satisfies

$$|g_{\mathrm{LW}}(\theta)|^2 = 1 - 4a^2\lambda^2(1 - a\lambda)\sin^4\left(\frac{\theta}{2}\right)$$

and is **dissipative of order 4** for $0 < |a\lambda| < 1$.

Crank-Nicolson, Leapfrog, Lax-Freidrichs

The leapfrog scheme and the Crank-Nicolson scheme are **strictly non-dissipative** since their amplification factors are identically 1 in magnitude.

$$g_{\text{CN}}(\theta) = \frac{1 - \frac{ia\lambda}{2}\sin(\theta)}{1 + \frac{ia\lambda}{2}\sin(\theta)}, \quad |g_{\text{CN}}(\theta)|^2 = \frac{1 + \frac{a^2\lambda^2}{4}\sin^2(\theta)}{1 + \frac{a^2\lambda^2}{4}\sin^2(\theta)} = 1.$$

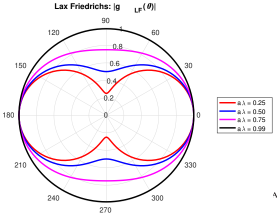
With $|a\lambda| < 1$ we have that

$$|g_{\text{Leap}}^{\pm}|^2 = 1 - (a\lambda)^2\sin^2(\theta) + (a\lambda)^2\sin^2(\theta) = 1,$$

The Lax-Friedrichs scheme is (non-strictly) non-dissipative

$$|g_{\text{LF}}(\theta)|^2 = \cos^2(\theta) + a^2\lambda^2\sin^2(\theta)$$

and $|g(\pi)| = 1$. This scheme will reduce the magnitude of most frequencies, but not the highest frequency on the grid.



Adding Dissipation to Schemes

It is possible to add dissipation to non-dissipative schemes, however care must be taken so that we do not affect the order of accuracy of the scheme.

The modified Leapfrog scheme given by

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} + \frac{\epsilon}{2k}\left[\frac{h\delta}{2}\right]^4 v_m^{n-1} = f_m^n$$

is second-order accurate for small values of ϵ, the corresponding amplification factor is

$$g_{\pm}(\theta) = -ia\lambda\sin(\theta) \pm \sqrt{1 - a^2\lambda^2\sin^2(\theta) - \epsilon\sin^4\left(\frac{\theta}{2}\right)}.$$

Adding Dissipation to Schemes

For small enough ϵ, the magnitude of the roots are given by

$$|g_{\pm}(\theta)|^2 = 1 - \epsilon \sin^4\left(\frac{\theta}{2}\right)$$

hence, the modified leapfrog scheme is dissipative of order 4.

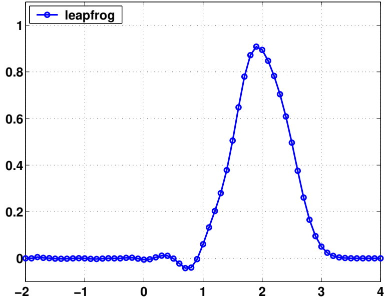


Figure: Comparison of [LEFT] modified leapfrog ($\epsilon = 0.5$), and [RIGHT] standard leapfrog; ($\lambda = 0.5$, $h = 0.1$).

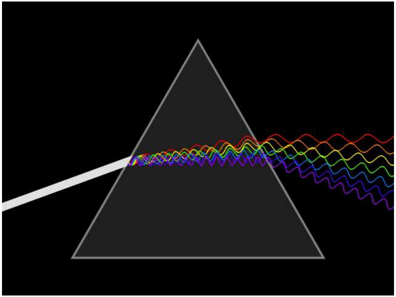


Figure: Physical Dispersion.

Dispersion: Introduction

We write the solution to the homogeneous one-way wave equation using the Fourier inversion formula

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega x} \underbrace{e^{-i\omega a t}\,\widehat{u}_0(\omega)}_{\widehat{u}(t,\omega)}\, d\omega,$$

from this representation we can identify

$$\widehat{u}(t+k,\omega) = e^{-i\omega a k}\,\widehat{u}(t,\omega).$$

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If we consider a **one-step** finite difference scheme, the propagation (in Fourier space) is given by

$$\widehat{v}^{n+1} = g(h\xi)\widehat{v}^{n}.$$

Clearly, if the scheme is good then $\mathbf{g(h\xi) \sim e^{-i\xi ak}}$.

Dispersion: Introduction

In order to clearly show the connection between the scheme, $g(h\xi)$, and the solution of the PDE, $e^{-i\omega ak}$, we write

$$g(h\xi) = |g(h\xi)| \, e^{-i\xi\alpha(\mathbf{h}\xi)k}$$

where $\alpha(h\xi)$ is interpreted as the **phase speed** — the speed at which waves of frequency ξ are propagated.

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If $\alpha(h\xi) \equiv a$, then all waves would propagate with correct speed. For many finite difference schemes **this does not hold**, and the difference $\mathbf{a} - \alpha(\mathbf{h}\xi)$ is known as the **phase error**.

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The **effect** of dispersion is shape-distortion of the solution.

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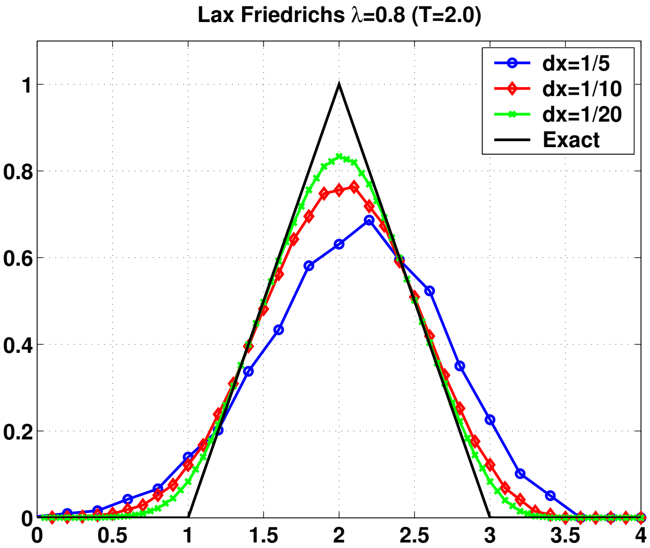
Dispersion ⇝ Shape Distortion



Dispersion Relations

With $\theta = h\xi$, we write

$$g(\theta) = |g(\theta)|\, e^{-i\theta\alpha(\theta)\lambda},$$

and dusting off some complex analysis, we identify

$$\tan\left[\alpha(\theta)\lambda\theta\right] = -\frac{\mathrm{Im}\,[g(\theta)]}{\mathrm{Re}\,[g(\theta)]}.$$

When $|g(\theta)| = 1$, we have

$$\sin\left(\,\alpha(\theta)\lambda\theta\,\right) = -\mathrm{Im}\,[g(\theta)]\,.$$

Example: The Lax-Wendroff Scheme

The amplification factor for the Lax-Wendroff scheme is

$$g(\theta) = 1 - 2(a\lambda)^2 \sin^2\left(\frac{\theta}{2}\right) - ia\lambda\sin(\theta)$$

By the preceding arguments we have,

$$\tan\left[\alpha(\theta)\lambda\theta\right] = \frac{a\lambda\sin(\theta)}{1 - 2(a\lambda)^2\sin^2\left(\frac{\theta}{2}\right)}$$

With a little help from our friend Taylor, this can give us some information about $a(\theta)$:

$$\begin{aligned}
\sin(\theta) &= \theta\left[1 - \frac{1}{6}\theta^2 + \mathcal{O}\left(\theta^4\right)\right] \\
\tan(\theta) &= \theta\left[1 + \frac{1}{3}\theta^2 + \mathcal{O}\left(\theta^4\right)\right] \\
\tan^{-1}(\theta) &= \theta\left[1 - \frac{1}{3}\theta^2 + \mathcal{O}\left(\theta^4\right)\right]
\end{aligned}$$

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Example: The Lax-Wendroff Scheme

We get

$$\alpha(\theta) = a\left[1 - \frac{1}{6}\theta^2\left(1-(a\lambda)^2\right) + \mathcal{O}\left(\theta^4\right)\right]$$

- When θ is small and $|a\lambda| < 1$, then $\alpha(\theta) < a$; the numerical solution will tend to trail the exact solution.
- When $|a\lambda| \approx 1$, then the dispersion (phase error) is smaller.

Example: The Lax-Wendroff Scheme

For larger values of θ, the Taylor expansions do not hold and we must consider the full expressions,

$$g(\theta) = 1 - 2(a\lambda)^2 \sin^2\left(\frac{\theta}{2}\right) - ia\lambda\sin(\theta),$$

and

$$\tan\Big[\alpha(\theta)\lambda\theta\Big] = \frac{a\lambda\sin(\theta)}{1 - 2(a\lambda)^2\sin^2\left(\frac{\theta}{2}\right)}.$$

When $\theta = \pi$, $g = 1 - 2a^2\lambda^2$:

- If $|a\lambda| > 1/\sqrt{2}$, then $g < 0$, and $\alpha(\pi) = 1/\lambda$
- If $|a\lambda| < 1/\sqrt{2}$, then $g > 0$, and $\alpha(\pi) = 0$

By consistency, $\alpha(0) = a$.

Example: The Lax-Wendroff Scheme

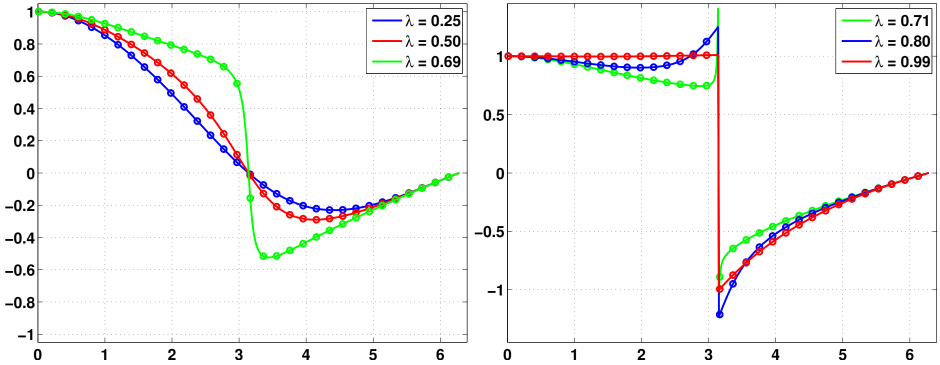

Figure: The phase speed for the Lax-Wendroff scheme; — [LEFT] $\lambda = \frac{1}{2}, \frac{1}{4}, 0.69 < \frac{1}{\sqrt{2}}$, AND [RIGHT] $\lambda = 0.99, 0.80, 0.71 > \frac{1}{\sqrt{2}}$; $(a = 1)$.

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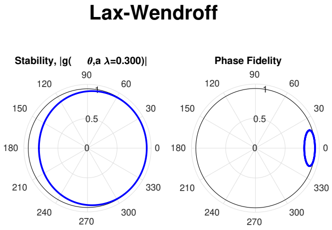
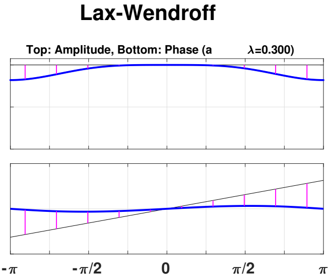
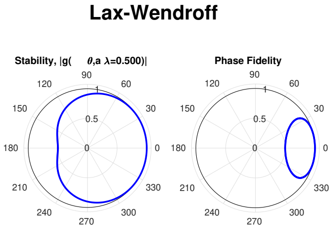
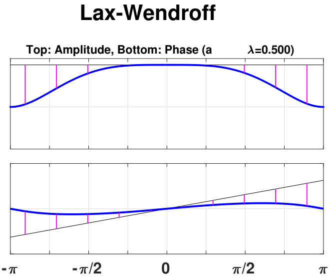
Example: The Lax-Wendroff Scheme









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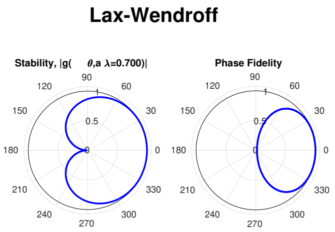
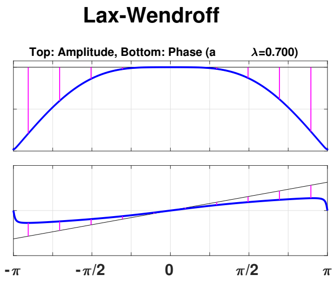
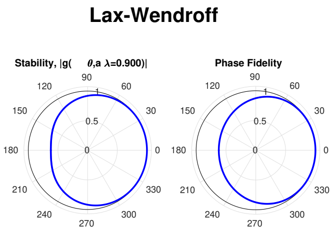
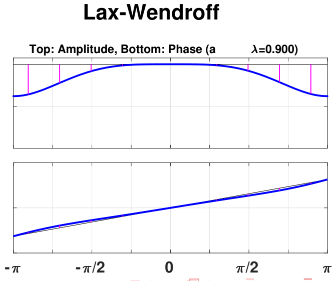
Example: The Lax-Wendroff Scheme









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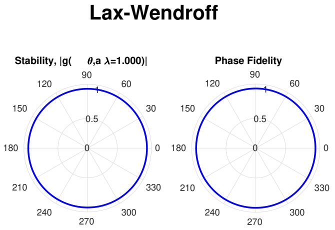

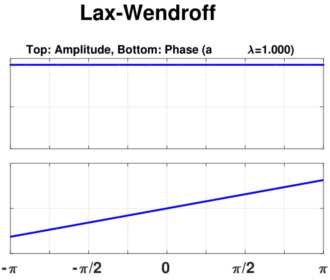

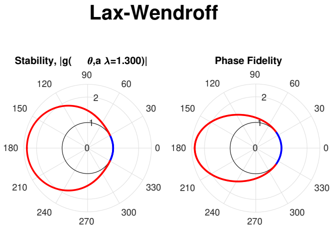

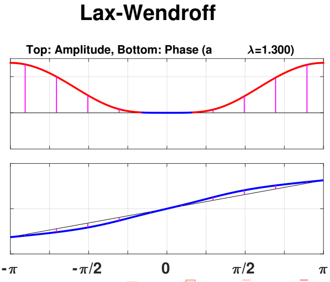

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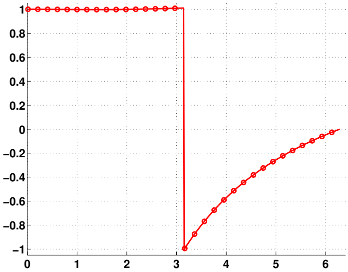
Rules of Thumb

- For hyperbolic PDEs is it best to take $|a\lambda|$ as close to the stability limit as possible; this keeps the dissipation and dispersion small, *e.g.* the phase speed for Lax-Wendroff for $a = 1$, and $\lambda = 0.99$:



If we are interested in a propagating a particular frequency ξ^*, then we must choose h so that $h\xi^* < \pi$. (**Think:** Nyquist sampling.)

Rules of Thumb

- When the leapfrog and Lax-Wendroff schemes are applied to the homogeneous one-way wave equation with $|a\lambda| = 1$, there is no dispersion error. These are exceptional special cases; when $a(t, x)$ is variable, or the system is non-trivial in any other way, there is dispersion.

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- The phase error is always an even function of θ, hence if a scheme has order of accuracy ρ, then the phase error is of order ρ if ρ is even, and order $\rho + 1$ if ρ is odd.

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- The phase error is always an even function of θ, hence if a scheme has order of accuracy ρ, then the phase error is of order ρ if ρ is even, and order $\rho + 1$ if ρ is odd.
- When choosing a scheme for a particular application, the amount of dissipation and dispersion can (and should) be used to choose between schemes, see *e.g.* D. Durran, *The Third-Order Adams-Bashforth Method: An Attractive Alternative to Leapfrog Time Differencing*, Monthly Weather Review, **119** (1991), pp. 702–720.

Wrapping Up, and Looking Forward

That ends our discussion of finite difference schemes for hyperbolic PDEs.

Next time we start looking at **parabolic PDEs**, of which the one-dimensional heat equation

$$\begin{aligned} \frac{\partial}{\partial t} u(t,x) &= b(t,x) \frac{\partial^2}{\partial x^2} u(t,x) \\ u(0,x) &= \varphi(x) \end{aligned}$$

is the simplest example.

The following result is fundamentally important in signal processing / information theory: —

Theorem (Sampling Theorem)

In order for a band-limited (i.e., one with a zero power spectrum for frequencies $\nu > B$) baseband ($\nu > 0$) signal to be reconstructed fully, it must be sampled at a rate $\nu \geq 2B$. A signal sampled at $\nu = 2B$ is said to be **Nyquist sampled**, *and $\nu = 2B$ is called the* **Nyquist frequency**. *No information is lost if a signal is sampled at the Nyquist frequency, and no additional information is gained by sampling faster than this rate.*

An audio CD is sampled at 44,100 samples/second, allowing for reconstruction of signals up to 22,050 Hz. (There is no evidence that human beings are sensitive to audio frequencies above 20 kHz, and most people over the age of 35 are unable to hear sounds above 15–16 kHz at 72 dB.)