

Numerical Solutions to PDEs

Lecture Notes #19 — Well-Posed and Stable Initial-Boundary Value Problems, Part 2

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Introduction

In the previous lecture we examined the well-posedness of a IBVP (on the PDE side), and the stability of the IBVP solved using the leapfrog scheme (on the finite difference side).

Our fundamental tool in this analysis is the **Laplace transform**

$$\tilde{u}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(\eta+i\tau)t} u(t) dt,$$

where $s = \eta + i\tau$.

As we saw in the leapfrog case, the analysis gets “a little” involved.

This time we state some general results for the stability analysis of boundary conditions and the well-posedness of the IBVP.



Outline

- 1 Introduction
 - Recap
- 2 Boundary Conditions: General Analysis
 - The Resolvent Equation
 - Characterizing Solutions
 - Stability of the Boundary Conditions
- 3 Examples
 - Crank-Nicolson for the One-Way Wave-Equation
 - Crank-Nicolson for the Heat Equation
- 4 Well-Posedness of the IBVP
 - General Solution and Boundary Condition
 - Admissible Solutions to the PDE
 - Well-Posed and Weakly Well-Posed IBVP



Boundary Conditions: General Analysis

We look the general method for checking the stability of boundary conditions for finite difference schemes.

We consider a scheme defined for all time and for $x \in \mathbb{R}^+$, with the boundary at $x = 0$:

$$P_{k,h} v_m^n = R_{k,h} f_m^n,$$

we assume the scheme is stable for the IVP and consistent with a hyperbolic equation (or system of equations, v_m^n is a d -vector); further we assume that no lower order terms are present (this simplifies the analysis). The boundary conditions are given by

$$B_{k,h} v_0^n = \beta(t_n),$$

For stability we must derive an estimate of the form

$$\eta \|v\|_{\eta,h}^2 + |v|_{\eta,h}^2 \leq C (\eta^{-1} \|f\|_{\eta,h}^2 + |\beta|_{\eta,h}^2 + \|v_0\|_h^2).$$



The Resolvent Equation

The Symbols Return

We Laplace ("z") transform $P_{k,h}v_m^n = 0$ in the t -direction ($v_m^n \rightsquigarrow z^n \tilde{v}_m$) and get, **the resolvent equation**

$$\tilde{P}_{k,h}(z)\tilde{v}_m(z) = 0,$$

the general solution is of the form

$$\tilde{v}_m(z) = A(z)\kappa^m, \tag{1}$$

which gives us $\tilde{P}_{k,h}(z)A(z)\kappa^m = k^{-1}\tilde{p}(z, \kappa)A(z)\kappa^m$, where the matrix function $\tilde{p}(z, \kappa)$ is related to the symbol of $P_{k,h}$, and the amplification polynomial by

$$\tilde{p}(e^{sk}, e^{ih\xi}) = kp_{k,h}(s, \xi), \quad \tilde{p}(g, e^{i\theta}) = \Phi(g, \theta).$$

Solutions of the form (1) exist only if $\det(\tilde{p}(z, \kappa)) = 0$, we view this as an equation for κ as a function of z .



Behavior of the Roots $\kappa(z)$...

Lemma

If $\kappa(z)$ is a root of $\det(\tilde{p}(z, \kappa)) = 0$ with $|\kappa(z)| = 1$ for $|z| = 1$, then there is a constant C such that

$$||\kappa| - 1| > C(|z| - 1)$$

whenever $|z| > 1$.



Behavior of the Roots $\kappa(z)$...

Theorem

If the scheme $P_{k,h}v_m^n = R_{k,h}f_m^n$ is stable, then there are integers K^- and K^+ such that the roots $\kappa(z)$, of $\det(\tilde{p}(z, \kappa)) = 0$ separate into two groups, one with K^- roots and one with K^+ roots. The group denoted by $\kappa_{-, \nu}(z)$ satisfy

$$|\kappa_{-, \nu}(z)| < 1 \quad \text{for } |z| > 1, \text{ and } \nu = 1, \dots, K^-$$

and the group denoted by $\kappa_{+, \nu}(z)$ satisfy

$$|\kappa_{+, \nu}(z)| > 1 \quad \text{for } |z| > 1, \text{ and } \nu = 1, \dots, K^+$$



Representation of the Solution

Characterization

By the previous theorem, K^- is **independent of z** , and the general solution in $L^2(\mathbb{R}^+)$ is given by

$$\tilde{v}_m(z) = \sum_{\nu=1}^{K^-} \alpha_\nu(z)A_\nu(z)\kappa_{-, \nu}^m.$$

Definition (Admissible Solutions)

An admissible solution to the resolvent equation is a solution that is in $L^2(h\mathbb{Z}^+)$ in the case when $|z| > 1$, and when $|z| = 1$ an admissible solution is the limit of admissible solutions with $|z| > 1$, i.e.

$$v(z) = \lim_{\epsilon \searrow 0} v(z(1 + \epsilon))$$

where $v(z(1 + \epsilon)) \in L^2(h\mathbb{Z}^+) \forall \epsilon > 0$.



The Coefficients $\alpha_\nu(z)$ and Stability...

As in the Leapfrog-case, the coefficients $\alpha_\nu(z)$ are determined by applying the Laplace transform to the boundary conditions

$$\tilde{B}\tilde{v}_0(z) = \tilde{\beta}(z).$$

The solution can be bounded independently of z **only if** there are no non-trivial solutions to the homogeneous equation for $|z| \geq 1$.

Thus checking for stability of the boundary conditions reduces to checking that there are **no admissible solutions** to the resolvent equation that also satisfy

$$\tilde{B}\tilde{v}_0(z) = 0.$$

We summarize this in a theorem:



Example #1: Crank-Nicolson for the One-Way Wave-Equation

We consider the Crank-Nicolson scheme applied to the one-way wave-equation $u_t + au_x = 0$

$$-\frac{a\lambda}{4}v_{m+1}^{n+1} + v_m^{n+1} + \frac{a\lambda}{4}v_{m-1}^{n+1} = \frac{a\lambda}{4}v_{m+1}^n + v_m^n - \frac{a\lambda}{4}v_{m-1}^n,$$

with quasi-characteristic extrapolation boundary condition

$$v_0^{n+1} = v_1^n.$$

Setting $\det(\tilde{\beta}(z, \kappa)) = 0$, gives us

$$\frac{z-1}{z+1} = \frac{a\lambda}{4} \left(\kappa - \frac{1}{\kappa} \right),$$

clearly if κ is a root, then so is $-\kappa^{-1}$ so that the roots $|\kappa_-(z)| < 1$ and $|\kappa_+(z)| > 1$ for $|z| > 1$, remain separated (as stated in the theorem on slide 6, with $K^- = K^+ = 1$).



Stability of the Boundary Conditions

Theorem (Stability of the Boundary Conditions)

The IBVP for the stable scheme $P_{k,h}v_m^n = R_{k,h}f_m^n$ for a **hyperbolic equation** with boundary conditions $B_{k,h}v_0^n = \beta(t_n)$ is stable **if and only if** there are no non-trivial solutions of the resolvent equation, $\tilde{P}_{k,h}(z)\tilde{v}_m(z) = 0$, that satisfy the homogeneous boundary conditions $\tilde{B}\tilde{v}_0(z) = 0$, for $|z| \geq 1$.

Theorem (Stability of the Boundary Conditions)

If the IBVP for the stable scheme $P_{k,h}v_m^n = R_{k,h}f_m^n$ with boundary conditions $B_{k,h}v_0^n = \beta(t_n)$ approximates a well-posed IBVP for a **parabolic PDE** and the number of boundary conditions required for the scheme is equal to the number required by the PDE, then the IBVP is stable **if and only if** there are no admissible solutions of the resolvent equation that satisfy the homogeneous boundary conditions for $|z| \geq 1$, **except for $z = 1$** .



Example #1: CN for the One-Way Wave-Equation

The boundary condition resulting from the substitution $\tilde{v}_m = \kappa_-^m$ (quasi-characteristic extrapolation) yields the equation

$$z - \kappa_-(z) = 0,$$

since $|z| \geq 1$, and $|\kappa_-(z)| \leq 1$, the only possible solution is $z = \kappa_-(z) = e^{i\theta}$, for some $\theta \in \mathbb{R}$. Thus we must have

$$\frac{e^{i\theta} - 1}{e^{i\theta} + 1} = \frac{a\lambda}{4} (e^{i\theta} - e^{-i\theta}),$$

or equivalently

$$\tan\left(\frac{\theta}{2}\right) = \frac{a\lambda}{2} \sin(\theta) = a\lambda \sin\left(\frac{\theta}{2}\right) \cos\left(\frac{\theta}{2}\right),$$

so that, either

$$\sin\left(\frac{\theta}{2}\right) = 0, \quad \text{or} \quad \cos^2\left(\frac{\theta}{2}\right) = \frac{1}{a\lambda}.$$



Example #1: CN for the One-Way Wave-Equation

3 of 4

Case #1, $\sin(\frac{\theta}{2}) = 0$: This is equivalent to $\kappa_-(1) = 1$, however as in the case of the leapfrog scheme $\lim_{\epsilon \searrow 0} [\kappa(1 + \epsilon)] \searrow 1$ i.e. $\kappa_+(1) = 1$, thus this case does not pose a difficulty.

Case #2, $\cos^2(\frac{\theta}{2}) = \frac{1}{a\lambda}$:

- (a) If $a\lambda < 1$, this does not have a solution.
- (b) If $a\lambda = 1$, $\cos^2(\frac{\theta}{2}) = 1$ only for $\theta = 0$, but as in case #1 this does not yield an admissible solution.
- (c) If $a\lambda > 1$, then we set

$$z = e^{i\theta} \frac{1 + \epsilon}{1 - \epsilon}, \quad \text{and} \quad \kappa = e^{i\theta}(1 + \delta)$$

and plug into

$$\frac{z - 1}{z + 1} = \frac{a\lambda}{4} \left(\kappa - \frac{1}{\kappa} \right)$$

With a little bit of help from Taylor, we get...



Example #1: CN for the One-Way Wave-Equation

4 of 4

...to first order in ϵ and δ

$$\epsilon \left(1 + \tan^2 \left(\frac{\theta}{2} \right) \right) = \frac{\delta a\lambda}{2} \cos(\theta)$$

so that if **(c-i)** $\cos(\theta) > 0$, then $\kappa_+ = z$, and if **(c-ii)** $\cos(\theta) < 0$, then $\kappa_- = z$ and the boundary condition is unstable. $\cos(\theta) < 0 \Leftrightarrow \cos^2(\frac{\theta}{2}) < \frac{1}{2}$, and therefore the scheme is unstable for $a\lambda > 2$.

Finally, for **(d)** $a\lambda = 2$ both $\kappa_- = \kappa_+ = z = \pm i$, and thus this case is also unstable. We conclude:

Stability Condition for the Boundary Condition

Case #1 and Case #2a-d show that the boundary condition is stable when $a\lambda < 2$.

Movies:

- [kappa_minus_alambda_0.5.mpg](#), [kappa_minus_alambda_1.5.mpg](#),
- [kappa_minus_alambda_2.0.mpg](#), [kappa_minus_alambda_2.5.mpg](#).



Note: $\cos(\theta) < 0 \Leftrightarrow \cos^2(\frac{\theta}{2}) < \frac{1}{2}$

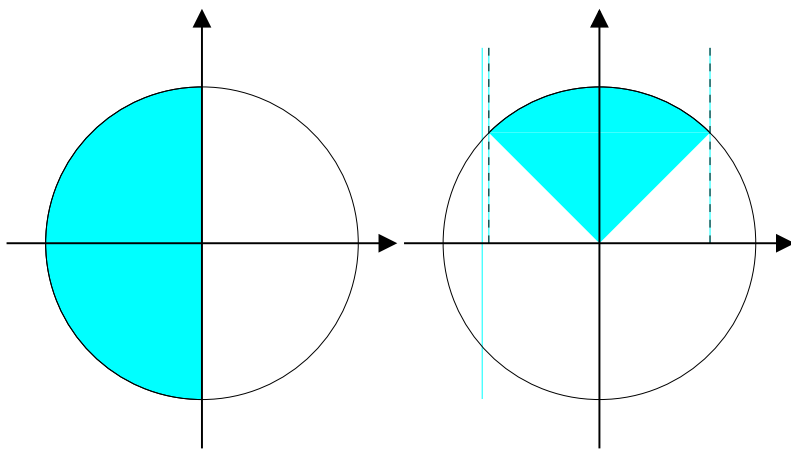


Figure: $\cos(\theta) < 0 \Leftrightarrow \theta \in (\pi/2, 3\pi/2) \Leftrightarrow \theta/2 \in (\pi/4, 3\pi/4) \Leftrightarrow \cos(\frac{\theta}{2}) \in (-1/\sqrt{2}, 1/\sqrt{2}) \Leftrightarrow \cos^2(\frac{\theta}{2}) \in [0, 1/2)$.



Example #2: Crank-Nicolson for the Heat Equation

1 of 2

Consider the heat equation $u_t = bu_{xx}$ on \mathbb{R}^+ with Neumann boundary condition $u_x = 0$ (no-flux) at $x = 0$, and the application of the Crank-Nicolson scheme

$$v_m^{n+1} - v_m^n = \frac{b\mu}{2} \delta^2 (v_m^{n+1} + v_m^n),$$

and the numerical implementation of the boundary condition

$$\frac{3v_0^{n+1} - 4v_1^{n+1} + v_2^{n+1}}{2h} = 0.$$

The equation relating κ and z is

$$\frac{z - 1}{z + 1} = b\mu(\kappa - 2 + \kappa^{-1}),$$

and the boundary condition gives

$$0 = 3 - 4\kappa_- + \kappa_-^2 = (1 - \kappa_-)(3 - \kappa_-).$$



Example #2: Crank-Nicolson for the Heat Equation

2 of 2

The boundary condition $(1 - \kappa_-)(3 - \kappa_-) = 0$ gives us $\kappa_- = 1$ as the only possibility with $|\kappa_-| \leq 1$. Plugging this into

$$\frac{z-1}{z+1} = b\mu \underbrace{(\kappa)}_1 - 2 + \underbrace{(\kappa^{-1})}_1 = 0$$

gives us $z = 1$.

This corresponds to the exception in the “parabolic theorem” (slide 10), and therefore the Finite Difference Scheme

$$v_m^{n+1} - v_m^n = \frac{b\mu}{2} \delta^2 (v_m^{n+1} + v_m^n),$$

with boundary condition

$$\frac{3v_0^{n+1} - 4v_1^{n+1} + v_2^{n+1}}{2h} = 0,$$

is stable.



General Solution and Boundary Condition

In the transform domain the general solution is given by

$$\hat{u}(s, x, \omega) = \hat{u}_0(s, \omega) e^{-\kappa x} + \frac{1}{2\kappa b} \int_x^\infty e^{(x-z)\kappa} \hat{f}(s, z, \omega) dz + \frac{1}{2\kappa b} \int_0^x e^{-(x-z)\kappa} \hat{f}(s, z, \omega) dz,$$

where $\kappa = \sqrt{b^{-1}s + \omega^2}$, and $\text{Re}(\kappa) > 0$, this gives the following characterization of the boundary condition

$$(-\kappa + i\alpha\omega) \left[\hat{u}_0(s, \omega) - \frac{1}{2\kappa} \int_0^\infty e^{-z\kappa} \hat{f}(s, z, \omega) dz \right] = \hat{\beta}(s, \omega),$$

this is a linear equation for \hat{u}_0 , which can only be solved if $(-\kappa + i\alpha\omega) \neq 0$, further if $|-\kappa + i\alpha\omega| \geq \delta > 0$, we can get a uniform estimate for \hat{u}_0 .



Well-Posedness of the IBVP

The remaining piece of the puzzle is a method for checking the well-posedness of the IBVP, as required in the theorem on slide 10.

On the domain $\Omega = \{(t, x, y) : t, y \in \mathbb{R}, x \in \mathbb{R}^+\}$, with boundary at $x = 0$, we consider the parabolic equation + boundary conditions

$$u_t = b(u_{xx} + u_{yy}) + f(t, x, y), \quad \text{Re}(b) > 0$$

$$u_x + \alpha u_y = \beta(t, y).$$

Fourier-transforming in y , and Laplace-transforming in t gives us

$$\hat{u}_{xx} = (b^{-1}s + \omega^2)\hat{u} - b^{-1}\hat{f}(s, x, \omega)$$

$$\hat{u}_x + i\omega\alpha\hat{u} = \hat{\beta}(s, \omega).$$



Condition for Well-Posedness

$-\kappa + i\alpha\omega = 0$ occurs only when

$$\sqrt{b^{-1}s + \omega^2} = i\alpha\omega \Leftrightarrow s = -b(\alpha^2 + 1)\omega^2$$

With $\text{Re}(s) \geq 0$ and ω real, this can only be satisfied if $\text{Re}[b(\alpha^2 + 1)] \leq 0$, thus **the requirement for the boundary condition $u_x + \alpha u_y = \beta(t, y)$ to be well-posed for equation $u_t = b(u_{xx} + u_{yy}) + f(t, x, y)$, $\text{Re}(b) > 0$ is**

$$\text{Re}[b(\alpha^2 + 1)] > 0.$$

Where

$$u_t = b(u_{xx} + u_{yy}) + f(t, x, y), \quad \text{Re}(b) > 0$$

$$u_x + \alpha u_y = \beta(t, y).$$



Observation and Generalization

The forcing function $f(t, x, y)$ does not impact the well-posedness of the boundary condition.

In general, for a PDE of the form

$$u_t = P(\partial_x, \partial_y)u + f(t, x, y)$$

$$Bu = \beta(t, y)$$

where $x \in \mathbb{R}^+$, $y \in \mathbb{R}^d$, the **resolvent equation** is an ODE for \hat{u}

$$[s - P(\partial_x, i\omega)]\hat{u} = 0, \quad \text{Re}(s) > 0$$

$$B\hat{u} = 0$$



Well-Posed IBVP

Theorem (Well-Posed IBVP)

The IBVP for $u_t = P(\partial_x, \partial_y)u + f(t, x, y)$ with boundary condition $Bu = \beta(t, y)$ is well-posed if and only if there are no non-trivial admissible solutions to the resolvent equation $[s - P(\partial_x, i\omega)]\hat{u} = 0$ that satisfy the homogeneous boundary conditions $B\hat{u} = 0$.

This theorem characterizes the strongest notion of a well-posed IBVP, involving estimates of the solution in the interior of the domain, as well as L^2 estimates of the solution on the boundary, in terms of the L^2 -norm of the boundary data.

A slightly relaxed version of the theorem turns out to be useful in applications (e.g. to CFD applications such as studying shallow water equations around a constant flow):



Admissible Solutions to the PDE

Definition (Admissible Solution)

An admissible solution to the resolvent equation

$$[s - P(\partial_x, i\omega)]\hat{u} = 0,$$

is a solution that is in $L^2(\mathbb{R}^+)$ as a function of x when $\text{Re}(s) > 0$, and, when $\text{Re}(s) = 0$ an admissible solution is the limit of admissible solutions with $\text{Re}(s) > 0$ positive, i.e.

$$\hat{u}(s, x, \omega) = \lim_{\epsilon \searrow 0} \hat{u}(s + \epsilon, x, \omega),$$

where $\hat{u}(s + \epsilon, x, \omega)$ is an admissible solution for each $\epsilon > 0$.



“Relaxed” Well-Posed IBVP: Weakly Well-Posed IBVP

Theorem (Weakly Well-Posed IBVP)

If a nontrivial admissible solution $\hat{u}(s_0, x, \omega_0)$ to the hyperbolic system $[s - P(\partial_x, i\omega)]\hat{u} = 0$ with $\text{Re}(s_0) = 0$, and $|s_0|^2 + |\omega_0|^2 \neq 0$ satisfies the homogeneous boundary condition $B\hat{u} = 0$, but there exists a constant c such that

$$\|B\hat{u}(s_0 + \epsilon, 0, \omega_0)\| \geq c\sqrt{\epsilon}\|\hat{u}(s_0, 0, \omega_0)\|$$

for $\epsilon > 0$ sufficiently small and there are no non-trivial solutions with $\text{Re}(s) > 0$ satisfying the homogeneous boundary conditions, then the IBVP is weakly well-posed.



Summarizing

[Lecture #18-19] We now have the tools to completely determined the well-posedness (or weak well-posedness) of an IBVP (on the model / PDE level), by establishing the non-existence of solutions to the resolvent equation, which we derive using Laplace-Fourier transforms.

On the computational (finite difference) level, the determination of stability for the boundary conditions follow a very similar pattern: discrete Laplace-Fourier \rightsquigarrow “discrete” resolvent equation...

[Lecture #17] We also have a quite complete picture of how the the smoothness of initial conditions affect the convergence rate of our schemes to the solution of the PDE.

[Lecture #16] In addition we have a clear characterization of well-posedness for initial value problems.

In the following lectures we expand our “problem universe” to include **elliptic problems**.

