Numerical Solutions to PDEs

Lecture Notes #19 — Well-Posed and Stable Initial-Boundary Value Problems, Part 2

Peter Blomgren,
⟨blomgren.peter@gmail.com⟩

Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

http://terminus.sdsu.edu/

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Outline

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Introduction

In the previous lecture we examined the well-posedness of a IBVP (on the PDE side), and the stability of the IBVP solved using the leapfrog scheme (on the finite difference side).

Our fundamental tool in this analysis is the **Laplace transform**

$$\tilde{u}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-(\eta+i\tau)t}\, u(t)\, dt,$$

where $s = \eta + i\tau$.

As we saw in the leapfrog case, the analysis gets "a little" involved.

This time we state some general results for the stability analysis of boundary conditions and the well-posedness of the IBVP.

Boundary Conditions: General Analysis

We look the general method for checking the stability of boundary conditions for finite difference schemes.

We consider a scheme defined for all time and for $x \in \mathbb{R}^+$, with the boundary at $x = 0$:

$$P_{k,h} v_m^n = R_{k,h} f_m^n,$$

we assume the scheme is stable for the IVP and consistent with a hyperbolic equation (or system of equations, v_m^n is a d-vector); further we assume that no lower order terms are present (this simplifies the analysis). The boundary conditions are given by

$$B_{k,h} v_0^n = \beta(t_n),$$

For stability we must derive an estimate of the form

$$\eta |\!|\!| v |\!|\!|_{\eta,h}^2 + |v|_{\eta,h}^2 \leq C \left(\eta^{-1} |\!|\!| f |\!|\!|_{\eta,h}^2 + |\beta|_{\eta,h}^2 + \| v_0 \|_h^2 \right).$$

The Resolvent Equation

The Symbols Return

We Laplace ("z") transform $P_{k,h} v_m^n = 0$ in the t-direction ($v_m^n \rightsquigarrow z^n \tilde{v}_m$) and get, **the resolvent equation**

$$\tilde{P}_{k,h}(z)\tilde{v}_m(z) = 0,$$

the general solution is of the form

$$\tilde{v}_m(z) = A(z)\kappa^m, \tag{1}$$

which gives us $\tilde{P}_{k,h}(z)A(z)\kappa^m = k^{-1}\tilde{p}(z,\kappa)A(z)\kappa^m$, where the matrix function $\tilde{p}(z,\kappa)$ is related to the symbol of $P_{k,h}$, and the amplification polynomial by

$$\tilde{p}(e^{sk}, e^{ih\xi}) = k p_{k,h}(s,\xi), \quad \tilde{p}(g, e^{i\theta}) = \Phi(g,\theta).$$

Solutions of the form (1) exist only if $\det(\tilde{\mathbf{p}}(\mathbf{z},\kappa)) = \mathbf{0}$, we view this as an equation for κ as a function of z.

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Behavior of the Roots $\kappa(z)$...

Theorem

If the scheme $P_{k,h}v_m^n = R_{k,h}f_m^n$ is stable, then there are integers K^- and K^+ such that the roots $\kappa(z)$, of $\det(\tilde{p}(z,\kappa)) = 0$ separate into two groups, one with K^- roots and one with K^+ roots. The group denoted by $\kappa_{-,\nu}(z)$ satisfy

$$|\kappa_{-,\nu}(z)| < 1 \quad \textit{for } |z| > 1, \textit{ and } \nu = 1, \ldots, K^-$$

and the group denoted by $\kappa_{+,\nu}(z)$ satisfy

$$|\kappa_{+,\nu}(z)| > 1 \quad \textit{for } |z| > 1, \textit{ and } \nu = 1, \ldots, K^+$$

Behavior of the Roots $\kappa(z)$...

Lemma

If $\kappa(z)$ is a root of $\det(\tilde{p}(z,\kappa)) = 0$ with $|\kappa(z)| = 1$ for $|z| = 1$, then there is a constant C such that

$$||\kappa| - 1| > C(|z| - 1)$$

whenever $|z| > 1$.

Representation of the Solution — Characterization

By the previous theorem, K^- is **independent of** z, and the general solution in $L^2(\mathbb{R}^+)$ is given by

$$\tilde{v}_m(z) = \sum_{\nu=1}^{K^-} \alpha_\nu(z) A_\nu(z) \kappa_{-,\nu}^m.$$

Definition (Admissible Solutions)

An admissible solution to the resolvent equation is a solution that is in $L^2(h\mathbb{Z}^+)$ in the case when $|z| > 1$, and when $|z| = 1$ an admissible solution is the limit of admissible solutions with $|z| > 1$, *i.e.*

$$v(z) = \lim_{\epsilon \searrow 0} v(z(1+\epsilon))$$

where $v(z(1+\epsilon)) \in L^2(h\mathbb{Z}^+)\ \forall \epsilon > 0$.

The Coefficients $\alpha_\nu(z)$ and Stability...

As in the Leapfrog-case, the coefficients $\alpha_\nu(z)$ are determined by applying the Laplace transform to the boundary conditions

$$\tilde{B}\tilde{v}_0(z) = \tilde{\beta}(z).$$

The solution can be bounded independently of z only if there are no non-trivial solutions to the homogeneous equation for $|z| \geq 1$.

Thus checking for stability of the boundary conditions reduces to checking that there are **no admissible solutions** to the resolvent equation that also satisfy

$$\tilde{B}\tilde{v}_0(z) = 0.$$

We summarize this in a theorem:

Stability of the Boundary Conditions

Theorem (Stability of the Boundary Conditions)

The IBVP for the stable scheme $P_{k,h}v_m^n = R_{k,h}f_m^n$ for a **hyperbolic equation** *with boundary conditions $B_{k,h}v_0^n = \beta(t_n)$ is stable if and only if there are no non-trivial solutions of the resolvent equation, $\tilde{P}_{k,h}(z)\tilde{v}_m(z) = 0$, that satisfy the homogeneous boundary conditions $\tilde{B}\tilde{v}_0(z) = 0$, for $|\mathbf{z}| \geq \mathbf{1}$.*

Theorem (Stability of the Boundary Conditions)

If the IBVP for the stable scheme $P_{k,h}v_m^n = R_{k,h}f_m^n$ with boundary conditions $B_{k,h}v_0^n = \beta(t_n)$ approximates a well-posed IBVP for a **parabolic PDE** *and the number of boundary conditions required for the scheme is equal to the number required by the PDE, then the IBVP is stable if and only if there are no admissible solutions of the resolvent equation that satisfy the homogeneous boundary conditions for $|\mathbf{z}| \geq \mathbf{1}$,* **except for $\mathbf{z} = \mathbf{1}$**.

Example #1: Crank-Nicolson for the One-Way Wave-Equation

We consider the Crank-Nicolson scheme applied to the one-way wave-equation $u_t + au_x = 0$

$$-\frac{a\lambda}{4} v_{m+1}^{n+1} + v_m^{n+1} + \frac{a\lambda}{4} v_{m-1}^{n+1} = \frac{a\lambda}{4} v_{m+1}^{n} + v_m^{n} - \frac{a\lambda}{4} v_{m-1}^{n},$$

with quasi-characteristic extrapolation boundary condition

$$v_0^{n+1} = v_1^n.$$

Setting $\det(\tilde{p}(z,\kappa)) = 0$, gives us

$$\frac{z-1}{z+1} = \frac{a\lambda}{4}\left(\kappa - \frac{1}{\kappa}\right),$$

clearly if κ is a root, then so is $-\kappa^{-1}$ so that the roots $|\kappa_-(z)| < 1$ and $|\kappa_+(z)| > 1$ for $|z| > 1$, remain separated (as stated in the theorem on slide 6, with $K^- = K^+ = 1$).

The boundary condition resulting from the substitution $\tilde{v}_m = \kappa_-^m$ (quasi-characteristic extrapolation) yields the equation

$$z - \kappa_-(z) = 0,$$

since $|z| \geq 1$, and $|\kappa_-(z)| \leq 1$, the only possible solution is $z = \kappa_-(z) = e^{i\theta}$, for some $\theta \in \mathbb{R}$. Thus we must have

$$\frac{e^{i\theta} - 1}{e^{i\theta} + 1} = \frac{a\lambda}{4}\left(e^{i\theta} - e^{-i\theta}\right),$$

or equivalently

$$\tan\left(\frac{\theta}{2}\right) = \frac{a\lambda}{2}\sin(\theta) = a\lambda\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right),$$

so that, either

$$\sin\left(\frac{\theta}{2}\right) = 0, \quad \text{or} \quad \cos^2\left(\frac{\theta}{2}\right) = \frac{1}{a\lambda}.$$

Example #1: CN for the One-Way Wave-Equation

Case #1, $\sin\left(\frac{\theta}{2}\right) = 0$**:** This is equivalent to $\kappa_-(1) = 1$, however as in the case of the leapfrog scheme $\lim_{\epsilon \searrow 0} [\kappa(1+\epsilon)] \searrow 1$ *i.e.* $\kappa_+(1) = 1$, thus this case does not pose a difficulty.

Case #2, $\cos^2\left(\frac{\theta}{2}\right) = \frac{1}{a\lambda}$**:**

(a) If $a\lambda < 1$, this does not have a solution.

(b) If $a\lambda = 1$, $\cos^2\left(\frac{\theta}{2}\right) = 1$ only for $\theta = 0$, but as in case #1 this does not yield an admissible solution.

(c) If $a\lambda > 1$, then we set

$$z = e^{i\theta}\frac{1+\epsilon}{1-\epsilon}, \quad \text{and} \quad \kappa = e^{i\theta}(1+\delta)$$

and plug into

$$\frac{z-1}{z+1} = \frac{a\lambda}{4}\left(\kappa - \frac{1}{\kappa}\right)$$

With a little bit of help from Taylor, we get...

...to first order in ϵ and δ

$$\epsilon\left(1+\tan^2\left(\frac{\theta}{2}\right)\right)=\frac{\delta a\lambda}{2}\cos(\theta)$$

so that if **(c-i)** $\cos(\theta) > 0$, then $\kappa_+ = z$, and if **(c-ii)** $\cos(\theta) < 0$, then $\kappa_- = z$ and the boundary condition is unstable. $\cos(\theta) < 0 \Leftrightarrow \cos^2\left(\frac{\theta}{2}\right) < \frac{1}{2}$, and therefore the scheme is unstable for $a\lambda > 2$. Finally, for **(d)** $a\lambda = 2$ both $\kappa_- = \kappa_+ = z = \pm i$, and thus this case is also unstable. We conclude:

Stability Condition for the Boundary Condition

Case #1 and **Case #2a–d** show that the boundary condition is stable when $a\lambda < 2$.

Movies:
kappa_minus_alambda_0.5.mpg, **kappa_minus_alambda_1.5.mpg**,
kappa_minus_alambda_2.0.mpg, **kappa_minus_alambda_2.5.mpg**.

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Note: $\cos(\theta) < 0 \Leftrightarrow \cos^2\left(\frac{\theta}{2}\right) < \frac{1}{2}$

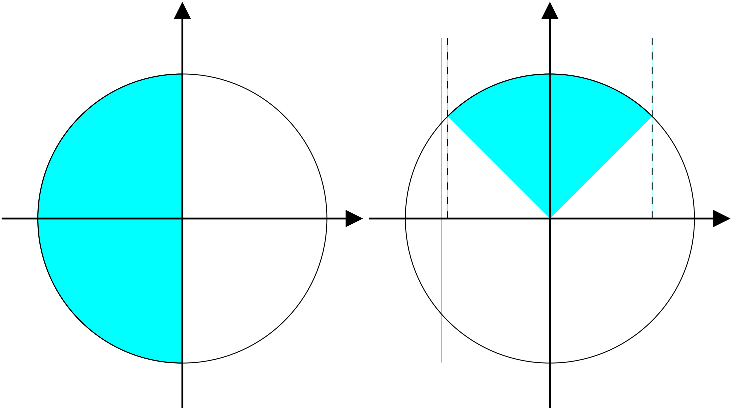

Figure: $\cos(\theta) < 0 \Leftrightarrow \theta \in (\pi/2, 3\pi/2) \Leftrightarrow \theta/2 \in (\pi/4, 3\pi/4) \Leftrightarrow \cos\left(\frac{\theta}{2}\right) \in (-1/\sqrt{2}, 1/\sqrt{2}) \Leftrightarrow \cos^2\left(\frac{\theta}{2}\right) \in [0, 1/2)$.

Example #2: Crank-Nicolson for the Heat Equation

Consider the heat equation $u_t = bu_{xx}$ on $\mathbb{R}^+$ with Neumann boundary condition $u_x = 0$ (no-flux) at $x = 0$, and the application of the Crank-Nicolson scheme

$$v_m^{n+1} - v_m^n = \frac{b\mu}{2}\delta^2(v_m^{n+1} + v_m^n),$$

and the numerical implementation of the boundary condition

$$\frac{3v_0^{n+1} - 4v_1^{n+1} + v_2^{n+1}}{2h} = 0.$$

The equation relating κ and z is

$$\frac{z-1}{z+1} = b\mu(\kappa - 2 + \kappa^{-1}),$$

and the boundary condition gives

$$0 = 3 - 4\kappa_- + \kappa_-^2 = (1 - \kappa_-)(3 - \kappa_-).$$

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Example #2: Crank-Nicolson for the Heat Equation

The boundary condition $(1-\kappa_-)(3-\kappa_-)=0$ gives us $\kappa_-=1$ as the only possibility with $|\kappa_-|\leq 1$. Plugging this into

$$\frac{z-1}{z+1} = b\mu(\underbrace{\kappa}_{1} - 2 + \underbrace{\kappa^{-1}}_{1}) = 0$$

gives us $z=1$.

This corresponds to the exception in the "parabolic theorem" (slide 10), and therefore the Finite Difference Scheme

$$v_m^{n+1} - v_m^n = \frac{b\mu}{2}\delta^2(v_m^{n+1}+v_m^n),$$

with boundary condition

$$\frac{3v_0^{n+1} - 4v_1^{n+1} + v_2^{n+1}}{2h} = 0,$$

is stable.

Well-Posedness of the IBVP

The remaining piece of the puzzle is a method for checking the well-posedness of the IBVP, as required in the theorem on slide 10.

On the domain $\Omega = \{(t,x,y) \,:\, t,y \in \mathbb{R},\, x \in \mathbb{R}^+\}$, with boundary at $x=0$, we consider the parabolic equation + boundary conditions

$$
\begin{aligned}
u_t &= b(u_{xx}+u_{yy}) + f(t,x,y), \quad \mathrm{Re}(b) > 0 \\
u_x + \alpha u_y &= \beta(t,y).
\end{aligned}
$$

Fourier-transforming in y, and Laplace-transforming in t gives us

$$
\begin{aligned}
\widehat{u}_{xx} &= (b^{-1}s + \omega^2)\widehat{u} - b^{-1}\widehat{f}(s,x,\omega) \\
\widehat{u}_x + i\omega\alpha\widehat{u} &= \widehat{\beta}(s,\omega).
\end{aligned}
$$

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General Solution and Boundary Condition

In the transform domain the general solution is given by

$$\widehat{u}(s,x,\omega) = \widehat{u}_0(s,\omega)e^{-\kappa x} + \frac{1}{2\kappa b}\int_x^\infty e^{(x-z)\kappa}\,\widehat{f}(s,z,\omega)\,dz + \frac{1}{2\kappa b}\int_0^x e^{-(x-z)\kappa}\,\widehat{f}(s,z,\omega)\,dz,$$

where $\kappa = \sqrt{b^{-1}s + \omega^2}$, and $\mathrm{Re}(\kappa) > 0$, this gives the following characterization of the boundary condition

$$(-\kappa \;+\; i\alpha\omega)\left[\widehat{u}_0(s,\omega) - \frac{1}{2\kappa}\int_0^\infty e^{-z\kappa}\,\widehat{f}(s,z,\omega)\,dz\right] \quad = \quad \widehat{\beta}(s,\omega),$$

this is a linear equation for $\widehat{u}_0$, which can only be solved if $(-\kappa + i\alpha\omega) \neq 0$, further if $|-\kappa + i\alpha\omega| \geq \delta > 0$, we can get a uniform estimate for $\widehat{u}_0$.

Condition for Well-Posedness

$-\kappa + i\alpha\omega = 0$ occurs only when

$$\sqrt{b^{-1}s + \omega^2} = i\alpha\omega \quad \Leftrightarrow \quad s = -b(\alpha^2+1)\omega^2$$

With $\mathrm{Re}(s) \geq 0$ and ω real, this can only be satisfied if $\mathrm{Re}\left[b(\alpha^2+1)\right] \leq 0$, thus **the requirement for the boundary condition** $u_x + \alpha u_y = \beta(t,y)$ **to be well-posed for equation** $u_t = b(u_{xx}+u_{yy}) + f(t,x,y),\ \mathrm{Re}(b) > 0$ **is**

$$\mathrm{Re}\left[\mathbf{b}(\alpha^{\mathbf{2}}+\mathbf{1})\right] > \mathbf{0}.$$

Where

$$\begin{aligned} u_t &= b(u_{xx}+u_{yy}) + f(t,x,y), \quad \mathrm{Re}(b) > 0 \\ u_x + \alpha u_y &= \beta(t,y). \end{aligned}$$

Observation and Generalization

The forcing function $f(t, x, y)$ does not impact the well-posedness of the boundary condition.

In general, for a PDE of the form

$$
\begin{aligned}
u_t &= P(\partial_x, \partial_y)u + f(t, x, y) \\
Bu &= \beta(t, y)
\end{aligned}
$$

where $x \in \mathbb{R}^+$, $y \in \mathbb{R}^d$, the **resolvent equation** is an ODE for $\widehat{u}$

$$
\begin{aligned}
[s - P(\partial_x, i\omega)]\widehat{u} &= 0, \quad \mathrm{Re}(s) > 0 \\
B\widehat{u} &= 0
\end{aligned}
$$

Admissible Solutions to the PDE

Definition (Admissible Solution)

An admissible solution to the resolvent equation

$$[s - P(\partial_x, i\omega)]\widehat{u} = 0,$$

is a solution that is in $L^2(\mathbb{R}^+)$ as a function of x when $\mathrm{Re}(s) > 0$, and, when $\mathrm{Re}(s) = 0$ an admissible solution is the limit of admissible solutions with $\mathrm{Re}(s) > 0$ positive, *i.e.*

$$\widehat{u}(s, x, \omega) = \lim_{\epsilon \searrow 0} \widehat{u}(s + \epsilon, x, \omega),$$

where $\widehat{u}(s + \epsilon, x, \omega)$ is an admissible solution for each $\epsilon > 0$.

Well-Posed IBVP

Theorem (Well-Posed IBVP)

The IBVP for $u_t = P(\partial_x, \partial_y)u + f(t,x,y)$ with boundary condition $Bu = \beta(t,y)$ is well-posed if and only if there are no non-trivial admissible solutions to the resolvent equation $[s - P(\partial_x, i\omega)]\widehat{u} = 0$ that satisfy the homogeneous boundary conditions $B\widehat{u} = 0$.

This theorem characterizes the strongest notion of a well-posed IBVP, involving estimates of the solution in the interior of the domain, as well as L^2 estimates of the solution on the boundary, in terms of the L^2-norm of the boundary data.

A slightly relaxed version of the theorem turns out to be useful in applications (*e.g.* to CFD applications such as studying shallow water equations around a constant flow):

"Relaxed" Well-Posed IBVP: Weakly Well-Posed IBVP

Theorem (Weakly Well-Posed IBVP)

If a nontrivial admissible solution $\widehat{u}(s_0, x, \omega_0)$ *to the hyperbolic system* $[s - P(\partial_x, i\omega)]\widehat{u} = 0$ *with* $\mathrm{Re}(s_0) = 0$*, and* $|s_0|^2 + |\omega_0|^2 \neq 0$ *satisfies the homogeneous boundary condition* $B\widehat{u} = 0$*, but there exists a constant c such that*

$$\|B\widehat{u}(s_0 + \epsilon, 0, \omega_0)\| \geq c\sqrt{\epsilon}\|\widehat{u}(s_0, 0, \omega_0)\|$$

for $\epsilon > 0$ *sufficiently small and there are no non-trivial solutions with* $\mathrm{Re}(s) > 0$ *satisfying the homogeneous boundary conditions, then the IBVP is weakly well-posed.*

Summarizing

[Lecture #18-19] We now have the tools to completely determined the well-posedness (or weak well-posedness) of an IBVP (on the model / PDE level), by establishing the non-existence of solutions to the resolvent equation, which we derive using Laplace-Fourier transforms.

On the computational (finite difference) level, the determination of stability for the boundary conditions follow a very similar pattern: discrete Laplace-Fourier $\rightsquigarrow$ "discrete" resolvent equation...

[Lecture #17] We also have a quite complete picture of how the the smoothness of initial conditions affect the convergence rate of our schemes to the solution of the PDE.

[Lecture #16] In addition we have a clear characterization of well-posedness for initial value problems.

In the following lectures we expand our "problem universe" to include **elliptic problems**.