Numerical Solutions to PDEs
Lecture Notes #2 — Finite Difference Schemes

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Outline

1. PDEs
   - Elliptic, Hyperbolic, and Parabolic

2. Hyperbolic PDEs
   - Introduction
   - Finite Difference Schemes
A second order PDE in two independent variables \((x, y)\) takes the form

\[ au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g \]

The coefficients \(a, b, c, d, e, f,\) and \(g\) are here (for now) assumed to be functions of \((x, y)\) only, so the equation is linear.

Through a change of variables

\[ \xi = \xi(x, y), \quad \eta = \eta(x, y) \]

it is possible to transform the PDE above to one of the three canonical forms (here the “…” terms hide (potentially) complicated expressions including \(u\) and its first derivatives):

\[ u_{\xi\xi} - u_{\eta\eta} + \cdots = 0, \]
\[ u_{\xi\xi} + \cdots = 0, \]
\[ u_{\xi\xi} + u_{\eta\eta} + \cdots = 0. \]
It can be shown that the coefficients for the second order term \((a, b \text{ and } c)\) in the PDE

\[ au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g \]

determine what canonical form the equation can be reduced to

<table>
<thead>
<tr>
<th>Canonical Form</th>
<th>Condition</th>
<th>Type</th>
</tr>
</thead>
<tbody>
<tr>
<td>(u_{\xi\xi} - u_{\eta\eta} + \cdots = 0)</td>
<td>(b^2 - ac &gt; 0)</td>
<td>Hyperbolic</td>
</tr>
<tr>
<td>(u_{\xi\xi} + \cdots = 0)</td>
<td>(b^2 - ac = 0)</td>
<td>Elliptic</td>
</tr>
<tr>
<td>(u_{\xi\xi} + u_{\eta\eta} + \cdots = 0)</td>
<td>(b^2 - ac &lt; 0)</td>
<td>Parabolic</td>
</tr>
</tbody>
</table>

**Examples:**
The Wave Equation is hyperbolic, the Heat Equation is parabolic, and Laplace’s equation is elliptic.
Three Main Types of PDEs

Rough characterizations:

- **Hyperbolic equations** have “wave-like” propagating solutions; where information propagates in space with finite speeds.

- **Parabolic equations** have “diffusion-like” solutions; where information gets “smoothed out” over time – the propagation speed may be infinite.

- **Elliptic equations** have no sense of “time evolution” and tend to show up in electrostatics, continuum mechanics, and as sub-problems in computational fluid dynamics.

- Many physical problems have multiple behaviors: imagine an oil-spill spreading out (diffusing) as it is being propagated by ocean currents.
We begin with an overview of Hyperbolic PDEs; from the simplest model equation, to hyperbolic systems, and equations with variable coefficients.

We introduce the central concepts convergence, consistency, and stability for finite difference schemes. These three concepts are related by the Lax-Richtmyer Theorem.
The full wave equation yields solutions propagating both ways; by formally “factoring” the differential operator
\[
\left( \partial_t^2 - a^2 \partial_x^2 \right) u = (\partial_t - a \partial_x) (\partial_t + a \partial_x) u \equiv (\partial_t + a \partial_x) (\partial_t - a \partial_x) u = 0,
\]
it is clear that solutions to either
\[
(\partial_t - a \partial_x) u = 0, \quad \text{or} \quad (\partial_t + a \partial_x) u = 0,
\]
are solutions to the original equation. These are known as \textbf{advection equations} describing a physical transport mechanism (with propagation speed \(a \text{ LENGTHUnits/TimeUnits}\)).
The simplest prototype for Hyperbolic PDEs is the one-way wave equation

\[ u_t(t, x) + au_x(t, x) = 0, \]

where \( a \) is a constant, \( t \in \mathbb{R}^+ \) represents time, and \( x \in \mathbb{R} \) the spatial location. The initial \( u(0, x) \) state must be specified.
Once the initial value $u(0, x) = u_0(x)$ is specified, the unique solution to the one-way wave equation for $t > 0$ is given by

$$u(t, x) = u_0(x - at).$$

The solution at time $t$ is just a shift of the initial value, $u_0(x)$. When $a > 0$ it is a shift to the right and when $a < 0$ it is a shift to the left.

The solution depends only on the value of $\xi = x - at$. These lines in the $(t, x)$-plane are called characteristics, and $\dim(a) = \dim(x) / \dim(t) = \text{length/time}$, hence $a$ is the propagation speed.

This is typical for Hyperbolic Equations: **The solution propagates with finite speed along characteristics.**
We note that the exact solution
\[ u(t, x) = u_0(x - at), \]
requires no differentiability of \( u \) (or \( u_0 \)), whereas the equation
\[ u_t + au_x = 0, \]
appears to only make sense if \( u \) is differentiable.

Hyperbolic equations feature solutions that are discontinuous (worse than non-differentiable); e.g. the \textbf{sonic boom} produced by an aircraft exceeding the speed of sound (Mach-1, or \( \approx 750 \) miles per hour at sea level) is an example of this phenomena. The discontinuity creates a \textbf{shock wave}.

Devising numerical schemes which allow for discontinuous solutions requires “a bit” of ingenuity.
A More General Hyperbolic Equation

\[ u_t + au_x + bu = f(t, x), \quad t > 0 \]
\[ u(0, x) = u_0(x) \]

Where \( a \) and \( b \) are constants. We can introduce the following change of variables (and its inverse):

\[
\begin{align*}
\tau &= t \\
\xi &= x - at,
\end{align*}
\]

With \( \tilde{u} (\tau, \xi) = u(t, x) \), we can transform the PDE to an ODE along the characteristics:

\[ \tilde{u}_\tau = -b \tilde{u} + f(\tau, \xi + a\tau). \]
The exact solution is given by
\[ \tilde{u}(\tau, \xi) = u_0(\xi)e^{-b\tau} + \int_0^\tau f(\sigma, \xi + a\sigma)e^{-b(\tau - \sigma)} \, d\sigma, \]
which expressed in the original variables is
\[ u(t, x) = u_0(x - at)e^{-bt} + \int_0^t f(s, x - a(t - s))e^{-b(t-s)} \, ds. \]

With some work this method can be extended to nonlinear equations of the form
\[ u_t + u_x = f(t, x, u), \quad \text{Note: } f \text{ depends on } u \]

From a numerical point of view, the key thing to note is that the solution evolves with **finite speed along the characteristics.**
Now consider systems of hyperbolic equations with constant coefficients in one space dimension; \( \mathbf{u} \) is now a \( d \)-dimensional vector (containing e.g. density, pressure, and velocity of a fluid or gas).

**Definition (Hyperbolic System)**

A system of the form

\[
\mathbf{u}_t + A\mathbf{u}_x + B\mathbf{u} = F(t, x)
\]

is hyperbolic if the matrix \( A \) is diagonalizable with real eigenvalues.
The matrix $A$ is **diagonalizable**, if there exists a non-singular matrix $P$ such that

$$PAP^{-1} = \text{diag}(\lambda_1, \ldots, \lambda_d) = \Lambda,$$

is a diagonal matrix. The eigenvalues $\lambda_1, \ldots, \lambda_d$ are the **characteristic speeds** of the system.

In the easiest case, $B = 0$, we get

$$\vec{w}_t + \Lambda \vec{w}_x = PF(t,x) = \tilde{F}(t,x)$$

under the change of variables $\vec{w} = P\vec{u}$. This is a reduction to $d$ independent scalar hyperbolic equations.

When $B \neq 0$, the resulting system is coupled, but only in undifferentiated terms. The lower order term $B\vec{u}$ causes growth, decay, or oscillations in the solution but **does not** alter the primary feature of solutions propagating along characteristics.
Example: Hyperbolic System

\[
\begin{bmatrix} u \\ v \end{bmatrix}_t + \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}_x = 0
\]

with \( u(0, x) = 1 \) if \(|x| \leq 1\), and 0 otherwise; and \( v(0, x) = 0 \).

The eigenvalues are \( \lambda = \{3, 1\} \), and without too much difficulty 

\( P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \) we can find the solution

\[
\begin{align*}
\ u(t, x) & = \frac{1}{2} [u_0(x - 3t) + u_0(x - t)], \\
\ v(t, x) & = \frac{1}{2} [u_0(x - 3t) - u_0(x - t)].
\end{align*}
\]
Example: Hyperbolic System

Figure: The solution at times $t = 0, 1/2, 1, 3/2, 2, 5/2$. (voir MOVIE)
Hyperbolic Equations with Variable Coefficients

What happens when the propagation speed is variable, e.g.

\[ u_t + a(t, x)u_x = 0 ? \]

In this example the solution is constant along characteristics, but the characteristics are not straight lines. Here, we get an ODE for the \( x \)-coordinate

\[ \frac{dx}{d\tau} = a(\tau, x), \quad x(0) = \xi. \]

If, e.g. \( a(\tau, x) = x \), then \( x(\tau) = \xi e^\tau \) (so that \( \xi = xe^{-t} \)), and we get

\[ u(t, x) = \tilde{u}(\tau, \xi) = u_0(\xi) = u_0(xe^{-t}). \]
We can extend the definition of hyperbolicity to systems:

**Definition (Hyperbolic System)**

A system of the form

\[ \bar{u}_t + A(t, x)\bar{u}_x + B(t, x)\bar{u} = F(t, x) \]

is hyperbolic if there exists a matrix function \( P(t, x) \) such that

\[ P(t, x)A(t, x)P^{-1}(t, x) = \text{diag}(\lambda_1(t, x), \ldots, \lambda_d(t, x)) = \Lambda(t, x) \]

is diagonal with real eigenvalues and the matrix norms of \( P(t, x) \) and \( P^{-1}(t, x) \) are bounded in \( x \) and \( t \) for \( x \in \mathbb{R}, t \geq 0 \).
Boundary Conditions

We now consider solving a hyperbolic equations on finite intervals, e.g. $0 \leq x \leq 1$.

First, consider the simple equation

$$u_t + au_x = 0, \quad 0 \leq x \leq 1, \quad t \geq 0,$$

If $a$ is positive then the information propagates to the right and if $a$ is negative it propagates to the left.
When $a > 0$, in addition to the initial value $u(0, x) \ 0 \leq x \leq 1$, we must also specify the boundary value $u(t, 0)$ for all $t > 0$, and when $a < 0$ we must specify $u(t, 1)$ for $t > 0$.

The problem of determining a solution when both initial and boundary data are present is known as an **Initial-Boundary Value Problem** (IBVP).
Consider the hyperbolic system (assume $a > 0$, $b > 0$)

\[
\begin{bmatrix}
  u \\
  v
\end{bmatrix}_t + \begin{bmatrix}
  a & b \\
  b & a
\end{bmatrix} \begin{bmatrix}
  u \\
  v
\end{bmatrix}_x = 0
\]

on the interval $0 \leq x \leq 1$. The characteristic speeds are $(a + b)$ and $(a - b)$, so that with $w = u + v$, and $z = u - v$

\[
\begin{bmatrix}
  w \\
  z
\end{bmatrix}_t + \begin{bmatrix}
  a + b & a - b \\
  a - b & a + b
\end{bmatrix} \begin{bmatrix}
  w \\
  z
\end{bmatrix}_x = 0
\]

If $b < a$, then both characteristic speeds are positive, but when $b > a$, we get one positive and one negative speed.
Figure: Illustration of Hyperbolic propagation; in the left panel $b < a$, so both characteristics propagate to the right. In the right panel $b > a$, so the characteristics propagate in opposite directions.
Figure: In order for the IBVPs to be well-posed we must (LEFT) specify the initial condition and two boundary conditions at $x = 0$; and (RIGHT) the initial condition, a boundary condition at $x = 0$, and a boundary condition at $x = 1$. Note that the specified boundary conditions must be linearly independent from the outgoing (leaving the domain) characteristic.
Let

\[ G(k, h) = \{(t_n, x_m) = (n \cdot k, m \cdot h) : n, m \in \mathbb{Z}\} \]

be a grid on \( \mathbb{R}^2 \):

We are interested in small values of \( h \), and \( k \) (sometimes denoted by \( \Delta x \), and \( \Delta t \); or \( \delta x \), and \( \delta t \)).
The basic idea is to replace derivatives by finite difference approximations, e.g. the time derivative at the point \((t_n, x_m)\) can be represented as

\[
\frac{\partial u}{\partial t}(t_n, x_m) \approx \begin{cases} 
\frac{u(t_n + k, x_m) - u(t_n, x_m)}{k} \\
\frac{u(t_n + k, x_m) - u(t_n - k, x_m)}{2k}
\end{cases}
\]

These are valid approximation since, for differentiable functions \(u\)

\[
\frac{\partial u}{\partial t}(t_n, x_m) = \begin{cases} 
\lim_{\epsilon \to 0} \frac{u(t_n + \epsilon, x_m) - u(t_n, x_m)}{\epsilon} \\
\lim_{\epsilon \to 0} \frac{u(t_n + \epsilon, x_m) - u(t_n - \epsilon, x_m)}{2\epsilon}
\end{cases}
\]

We frequently use the notation \(v_n^m = u(t_n, x_m)\).
Applying these ideas to \( u_t + au_x = 0 \) we can write down a number of finite difference approximations at \((t_n, x_m)\), e.g.

\[
\frac{v_{m}^{n+1} - v_{m}^{n}}{k} + a \frac{v_{m+1}^{n+1} - v_{m}^{n}}{h} = 0 \quad \text{Forward-Time-Forward-Space}
\]

\[
\frac{v_{m}^{n+1} - v_{m}^{n}}{k} + a \frac{v_{m}^{n} - v_{m-1}^{n}}{h} = 0 \quad \text{Forward-Time-Backward-Space}
\]

\[
\frac{v_{m}^{n+1} - v_{m}^{n}}{k} + a \frac{v_{m+1}^{n+1} - v_{m-1}^{n}}{2h} = 0 \quad \text{Forward-Time-Central-Space}
\]

\[
\frac{v_{m}^{n+1} - v_{m}^{n-1}}{2k} + a \frac{v_{m+1}^{n+1} - v_{m-1}^{n}}{2h} = 0 \quad \text{Central-Time-Central-Space, leapfrog}
\]

It is quite easy to derive these schemes (see polynomial approximation in Math 541) and/or to see that they may be viable approximations.
The main difficulty of finite difference schemes is the analysis required to determine if they are **useful approximations**. Indeed, some of the schemes on the previous slide are useless.

The schemes presented so far can all be written expressing $v_{m}^{n+1}$ as linear combinations of $v_{\mu}^{\nu}$ at previous time-levels $\nu \in \{n-1, n\}$. The Forward-Time-Forward-Space scheme can be written as

$$v_{m}^{n+1} = (1 + a\lambda)v_{m}^{n} - a\lambda v_{m+1}^{n}$$

where $\lambda = k/h$ is the ratio of the time- and space- discretization. This scheme is a **one-step scheme** since it only involves information from one previous time-level.

The leapfrog scheme is a two-step (multi-step) scheme.
Example: Leapfrog Solutions

Figure: Solutions for the leapfrog scheme with $\lambda = \{0.8, 0.95, 1.02\}$ for the equation $u_t + u_x = 0$ with initial condition

$$u_0(x) = \begin{cases} 
1 - |x| & \text{if } |x| \leq 1 \\
0 & \text{otherwise}
\end{cases}$$

and boundary condition

$$u(t, -2) = 0.$$

Clearly something “strange” happens when we let $\lambda > 1$. We introduce the discussion on convergence, consistency, and stability next time. (∃Movie)
**Figure:** We notice that the errors shrink with the size of $dx$ when $\lambda < 1$, but grow when $\lambda > 1$. 

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Lecture Notes #2 — Finite Difference Schemes — (29/29)