Math 254: Introduction to Linear Algebra
Lecture Notes #1.1 — Linear Equations

Peter Blomgren,
⟨blomgren.peter@gmail.com⟩

Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

http://terminus.sdsu.edu/

Fall 2017
1 Student Learning Objectives
   - SLOs: Linear Equations
   - Numbering of Lecture Notes

2 Linear Equations
   - Example — Finding the Unique Solution
   - A Case with Infinitely Many Solutions
   - A Case with No Solution

3 Suggested Problems
   - Suggested Problems 1.1
After this lecture you should:

- Be able to *Interpret* solutions of linear systems of 3 variables as intersections of planes.
- *Know* that linear systems can have no, one, or infinitely many solutions.
- Be able to *Perform* basic row-operations to determine all solutions of linear systems.
Why are the Notes Numbered the Way They Are???

The numbering of the topics originates from the structure of Otto Bretcher’s book (used Fall 2015 — Fall 2016).

Since adopting Gilbert Strang’s book (Spring 2017 —) the main “chapter topic number” (the “1” in 1.n) is retained, but the “section number” the (“1” in n.1) is slowly being replaced by the enumerated lecture number on each topic.

This is the first lecture on the First topic (Linear Equations); hence Notes #1.1.

References to the particular sections of Gilbert Strang’s book will be added in the form [GS5–§1.1] (meaning section 1.1 in the 5th edition).
A First Example

We sweep the history lessons under my infinitely stretchable rug, and focus on the system of linear equations:

\[
\begin{align*}
x + 2y + 3z &= 39 & r_1 \quad \text{"row #1"} \\
x + 3y + 2z &= 34 & r_2 \\
3x + 2y + z &= 26 & r_3
\end{align*}
\]

and the question:

— What values of \( x \), \( y \), and \( z \) satisfy this system?
We want to manipulate the system

\[
\begin{align*}
\text{From} & \quad x + 2y + 3z &= 39 \\
& \quad x + 3y + 2z &= 34 \\
& \quad 3x + 2y + z &= 26
\end{align*}
\]

\[
\begin{align*}
\text{To} & \quad x &= ??? \\
& \quad y &= ??? \\
& \quad z &= ???
\end{align*}
\]

But, before we do that, let’s discuss a Graphical / Geometric Interpretation of the system...
If we view each row/equation in the system

\[
\begin{align*}
    x + 2y + 3z &= 39 \\
    x + 3y + 2z &= 34 \\
    3x + 2y + z &= 26
\end{align*}
\]

as a plane in 3D (x-y-z) space; the solution represents the point(s?) where the planes meet.

Solve for \( z \) in each one of the equations:

\[
\begin{align*}
    z_1(x, y) &= (39 - x - 2y)/3 \\
    z_2(x, y) &= (34 - x - 3y)/2 \\
    z_3(x, y) &= (26 - 3x - 2y)
\end{align*}
\]
The Three Planes

**Figure:** The planes $z_1(x, y)$, $z_2(x, y)$, and $z_3(x, y)$ visualized. As a reference, the black lines on the planes are aligned with the $x$- and $y$-axes.
The Three Planes — Intersecting

**Figure:** The planes $z_1(x, y)$, $z_2(x, y)$, and $z_3(x, y)$ visualized. We are looking for the (in this case) ONE POINT where they all meet. [ ≡ Movie]
OK, Back to Solving the System

We can ADD or SUBTRACT multiples (fractions) of any equation (row) to/from another equation; or MULTIPLY / DIVIDE / SCALE an equation, without changing the solution.

We do this in order to successively eliminate variables from the equations, so that in the end we reveal the solution.

[GS5–§2.2 — “The Idea of elimination”]
“Forward elimination” stage:

\[
\begin{align*}
  x + 2y + 3z &= 39 \\
  x + 3y + 2z &= 34 \\
  3x + 2y + z &= 26
\end{align*}
\]

we get:

\[
\begin{align*}
  x + 2y + 3z &= 39 \\
  y - z &= -5 \\
  -4y - 8z &= -91
\end{align*}
\]

\[
\begin{align*}
  x + 2y + 3z &= 39 \\
  y - z &= -5 \\
  -12z &= -111
\end{align*}
\]

divide by \((-12)\)
“Backward elimination” or “Back-substitution” starts...

\[
\begin{align*}
  x + 2y + 3z &= 39 & \text{subtract } 3r_3 \\
  y - z &= -5 & \text{add } r_3 \\
  z &= 9.25
\end{align*}
\]

\[
\begin{align*}
  x + 2y &= 11.25 & \text{subtract } 2r_2 \\
  y &= 4.25 \\
  z &= 9.25
\end{align*}
\]

\[
\begin{align*}
  x &= 2.75 \\
  y &= 4.25 \\
  z &= 9.25
\end{align*}
\]
Check the Solution!

Plug

<table>
<thead>
<tr>
<th>x</th>
<th>= 2.75</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y)</td>
<td>= 4.25</td>
</tr>
<tr>
<td>(z)</td>
<td>= 9.25</td>
</tr>
</tbody>
</table>

into

\[
\begin{align*}
x + 2y + 3z &= 39 \\
x + 3y + 2z &= 34 \\
3x + 2y + z &= 26
\end{align*}
\]

and see that

\[
\begin{align*}
2.75 + 2 \times 4.25 + 3 \times 9.25 &= 39 \\
2.75 + 3 \times 4.25 + 2 \times 9.25 &= 34 \\
3 \times 2.75 + 2 \times 4.25 + 9.25 &= 26
\end{align*}
\]
There are different paths to the solution; for instance you can fully eliminate each column ("up" as well as "down") before proceeding to the next one; in the end you will arrive at the same solution. (This strategy does not separate into "forward" and "backward" stage)
When you do things by “hand” you will probably pick a path (order of operations) which is unique to the problem, and which makes the algebra as simple as possible.

When you do things by “code” (in software), you develop an algorithm (recipe / step-by-step instructions) which always follows the same path, since no algebra is “hard” for the computer (except dividing by zero... which is very very bad.) — My brain is almost “software” by now, so I like to think of the process in terms of a forward “sweep” followed by a backward “sweep.”

The forward/backward split has the added benefit that collecting the results gives some useful “side products” — see [GS5–§2.6 — “Elimination = Factorization; \( A = LU \)’]
In a lot of books the Geometric Interpretation “lives” as its own separate “thing,” but it is natural to wonder what the impact of the solution procedure (Elimination) has on the Geometry.

\[
\begin{align*}
  x + 2y + 3z &= 39 \\
  x + 3y + 2z &= 34 \\
  3x + 2y + z &= 26
\end{align*}
\]

Peter Blomgren, ⟨blomgren.peter@gmail.com⟩

Lecture Notes #1.1 — Linear Equations
Eliminate $x$ from Equations 2, and 3

\[
\begin{align*}
x + 2y + 3z &= 39 \\
y - z &= -5 \\
3x + 2y + z &= 26
\end{align*}
\]

\[
\begin{align*}
x + 2y + 3z &= 39 \\
y - z &= -5 \\
-4y - 8z &= -91
\end{align*}
\]

Note: the new $z_2(x, y) = y + 5$, $z_3(x, y) = (91 - 4y)/8$ do not depend on $x$.
Eliminate $y$ from Equation 3; and $z$ from Equation 2

\[
\begin{align*}
    x + 2y + 3z &= 39 \\
    y - z &= -5 \\
    z &= 9.25
\end{align*}
\]

\[
\begin{align*}
    x + 2y + 3z &= 39 \\
    y &= 4.25 \\
    z &= 9.25
\end{align*}
\]
Example — Finding the Unique Solution

A Case with Infinitely Many Solutions

A Case with No Solution

Eliminate $z$ from Equation 1; and $y$ from Equation 1

\[
\begin{align*}
\begin{array}{l}
x + 2y = 11.25 \\
y = 4.25 \\
z = 9.25
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{l}
x = 2.75 \\
y = 4.25 \\
z = 9.25
\end{array}
\end{align*}
\]
Looking the sequence of intersection planes as we go through the elimination steps:

We realize that our geometric goal was to create orthogonal planes intersecting in the solution point.

The idea of orthogonalization will show up in various contexts; but it is always a tool which makes it easy (well, easier) to identify the property we are after. — In this particular case the solution to a linear system.
What Can Happen?

A Linear System can have

- a unique solution (like our previous example)
- infinitely many solutions
- no solution
More solutions than you can shake an infinite stick at...

\[
\begin{align*}
2x + 4y + 6z &= 0 \\
4x + 5y + 6z &= 3 \\
7x + 8y + 9z &= 6
\end{align*}
\]

**Figure:** We sense trouble when the three planes meet not in one point, but along a common line in space. Let us see how that manifests itself in the elimination process.
Looking for infinitely many solutions...

\[
\begin{align*}
2x + 4y + 6z &= 0 \\
4x + 5y + 6z &= 3 \\
7x + 8y + 9z &= 6
\end{align*}
\]

- Divide by 2
- Divide by 3

\[
\begin{align*}
x + 2y + 3z &= 0 \\
4x + 5y + 6z &= 3 \\
7x + 8y + 9z &= 6
\end{align*}
\]

- Subtract 4 times the first equation
- Subtract 7 times the first equation

\[
\begin{align*}
x + 2y + 3z &= 0 \\
-3y - 6z &= 3 \\
-6y - 12z &= 6
\end{align*}
\]

- Divide by -3
- Divide by -6
Looking for infinitely many solutions...

\[
\begin{align*}
\begin{bmatrix}
x + 2y + 3z &= 0 \\
y + 2z &= -1 \\
y + 2z &= -1 \\
\end{bmatrix} &\rightarrow -r_2 \\
\begin{bmatrix}
x + 2y + 3z &= 0 \\
y + 2z &= -1 \\
0 &= 0 \\
\end{bmatrix} &\rightarrow -2r_2 \\
\begin{bmatrix}
x - z &= 2 \\
y + 2z &= -1 \\
0 &= 0 \\
\end{bmatrix}
\end{align*}
\]
We have:

\[
\begin{align*}
  x - z &= 2 \\
  y + 2z &= -1 \\
  0 &= 0
\end{align*}
\]

This means that

\[
\begin{align*}
  x &= 2 + z \\
  y &= -1 - 2z
\end{align*}
\]

describes the line in space where the planes intersect; \(x\) and \(y\) are given as functions of \(z\); the line is \((2 + z, -1 - 2z, z)\), \(z \in [-\infty, \infty]\).

**Note:** We can write this as \((x, y, z) = (2, -1, 0) + t(1, -2, 1),\) \(t \in [-\infty, \infty]\).
Somebody ran away with the solution!

\[
\begin{align*}
  x + 2y + 3z &= 0 \\
  4x + 5y + 6z &= 3 \\
  7x + 8y + 9z &= 0
\end{align*}
\]

**Figure:** Here, the planes intersect (pair-wise), but not in any common point, or line. Again, we go through the elimination to see how this manifests itself in the computation.
Looking for Nothing...

\[
\begin{align*}
  x + 2y + 3z &= 0 \\
  4x + 5y + 6z &= 3 & \rightarrow -4r_1 \\
  7x + 8y + 9z &= 0 & \rightarrow -7r_1 \\
  x + 2y + 3z &= 0 \\
  -3y - 6z &= 3 & \text{divide by } -3 \\
  -6y - 12z &= 0 & \text{divide by } -6 \\
  x + 2y + 3z &= 0 \\
  y + 2z &= -1 \\
  y + 2z &= 0 & \rightarrow -r_2
\end{align*}
\]
Looking for Nothing... and Finding Nonsense!

\[
\begin{align*}
  x + 2y + 3z &= 0 \\
y + 2z &= -1 \\
0 &= 1
\end{align*}
\]

Say "What?!?"

This system of equations is said to be \textit{inconsistent}, and no solutions exist.

\textbf{However:} It is still possible to find the pair-wise intersections of the planes. Since the first 2 equations are the same as in the previous $\infty$-many solutions example; one such line intersection is 
$(x, y, z) = (2, -1, 0) + t(1, -2, 1), \ t \in [-\infty, \infty]$. 

Peter Blomgren, \{blomgren.peter@gmail.com\}
Finding More Pair-Wise Intersections (#2)

If we grab equations #1 and #3:

$$\begin{align*}
  x + 2y + 3z &= 0 \\
  7x + 8y + 9z &= 0 \\
  -6y - 12z &= 0
\end{align*}$$

-divide by $-6$

$$\begin{align*}
  x + 2y + 3z &= 0 \\
  -2r_2 \\
  y + 2z &= 0
\end{align*}$$

Giving the line $(x, y, z) = (0, 0, 0) + t(1, -2, 1), \ t \in [-\infty, \infty]$. 
Finally, we grab equations #2 and #3:

\[
\begin{align*}
4x + 5y + 6z &= 3 \\
7x + 8y + 9z &= 0
\end{align*}
\]

\[
\begin{align*}
x + \frac{5}{4}y + \frac{6}{4}z &= \frac{3}{4} \\
x + \frac{8}{7}y + \frac{9}{7}z &= 0
\end{align*}
\]

\[
\begin{align*}
x + \frac{5}{4}y + \frac{6}{4}z &= \frac{3}{4} \\
-\frac{3}{28}y - \frac{3}{14}z &= -\frac{3}{4}
\end{align*}
\]

\[
\begin{align*}
\frac{1}{4} & \quad \frac{1}{7} \\
-\frac{1}{28} & \quad -\frac{1}{14}
\end{align*}
\]
Finding More Pair-Wise Intersections (#3)

\[
\begin{vmatrix}
  x & - & z & = & -8 \\
  - & \frac{3}{28}y & - & \frac{3}{14}z & = & -\frac{3}{4} \\
  x & - & z & = & -8 \\
  y & + & 2z & = & 7
\end{vmatrix}
\]

Giving the line \((x, y, z) = (-8, 7, 0) + t (1, -2, 1), \ t \in [-\infty, \infty].\)

Leaving us with 3 lines in space

\[
\begin{align*}
(x, y, z)_1 &= (2, -1, 0) + t(1, -2, 1) \\
(x, y, z)_2 &= (0, 0, 0) + t(1, -2, 1) \\
(x, y, z)_3 &= (-8, 7, 0) + t (1, -2, 1)
\end{align*}
\]
Pair-Wise Intersections — Visualized

\[ (x, y, z)_1 = (2, -1, 0) + t(1, -2, 1) \]
\[ (x, y, z)_2 = (0, 0, 0) + t(1, -2, 1) \]
\[ (x, y, z)_3 = (-8, 7, 0) + t(1, -2, 1) \]

**Figure:** The lines are parallel, and never intersect.

OK, we have squeezed the last “juice” out of this problem...
Available on Learning Glass videos:
1.1 — 1, 3, 7, 14, 19, 21, 42

Not (yet?) on Learning Glass:
GS5 — 2.2: 1, 2, 12, 19,