Outline

1 Student Learning Objectives
   - SLOs: Matrices, Vectors, ...

2 Matrices, Vectors; Gauss-Jordan Elimination
   - Matrix – Vector Notation
   - Back to Solving Linear Systems
   - Summarizing

3 Suggested Problems
   - Suggested Problems
   - Lecture – Book Roadmap

4 Supplemental Material
   - Metacognitive Reflection
   - Problem Statements 1.2

Equation Manipulation to Isolate Variables

After this lecture you should:

- Know basic language and concepts:
  - Matrices, vectors, and their components
  - Matrix types: square, diagonal, triangular, zero, identity
  - The collection of all \( n \)-vectors, denoted \( \mathbb{R}^n \) is a vector space

- Vector addition

- Know the difference between the Coefficient matrix, and the Augmented matrix; their uses in the solution of linear systems

- Know how to use elimination to identify leading and non-leading (a.k.a. free variables), and when necessary introduce parameters to express all solutions of linear systems.

- Know what Reduced–Row–Echelon–Form (RREF) of a Matrix is, and how to achieve it using elementary row operations.

Our current “business” is manipulating linear systems, of the form

\[
\begin{bmatrix}
3x & + & 21y & - & 3z & = & 0 \\
-6x & - & 2y & - & z & = & 62 \\
2x & - & 3y & + & 8z & = & 32
\end{bmatrix}
\]

into a form which reveals the values of \( x \), \( y \), and \( z \):

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} = \begin{bmatrix}
-3574/281 \\
844/281 \\
2334/281
\end{bmatrix}
\]

We achieve this by cleverly adding/subtracting rows (equations) from each other.
Matrix Notation — “Encoding” the Information

We realize that the important information is in the coefficients (numbers), and that the variables \((x, y, z)\) just get carried around. We can “encode” all the information about the linear system

\[
\begin{align*}
3x + 21y - 3z &= 0 \\
-6x - 2y - z &= 62 \\
2x - 3y + 8z &= 32
\end{align*}
\]

in a **matrix**

\[
\begin{bmatrix}
3 & 21 & -3 & 0 \\
-6 & -2 & -1 & 62 \\
2 & -3 & 8 & 32
\end{bmatrix}
\]

or sometimes:

\[
\begin{bmatrix}
3 & 21 & -3 & 0 \\
-6 & -2 & -1 & 62 \\
2 & -3 & 8 & 32
\end{bmatrix}
\]

Augmented Matrix

Ponder the matrix “\(A\)” with 3 rows, and 4 columns:

\[
A = \begin{bmatrix}
3 & 21 & -3 & 0 \\
-6 & -2 & -1 & 62 \\
2 & -3 & 8 & 32
\end{bmatrix}
= \begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34}
\end{bmatrix}
\]

that is, usually we refer to the entries of a matrix \(A\) (upper-case), using double subscripts \(a_{ij}\) (lower-case); the subscripts \(i\), and \(j\) are “standard” but \(r\) (row) and \(c\) (column) would be more intuitive.

Sometimes you see the notation \(A \in \mathbb{R}^{3 \times 4}\) to denote a 3-by-4 (always \([\text{RO}w-\text{BY-}C\text{O}L\text{U}M\text{N}]\)) matrix where the entries are real \((\mathbb{R})\) numbers.

**Note:** The entries can be other mathematical objects, e.g. complex numbers, \(\mathbb{C}\), polynomials, etc... but we will work with \(\mathbb{R}\) for quite while.

Row – Column Indexing

From a computer science point-of-view a matrix can be viewed as a **data structure**, and depending on your mood (and/or preference of programming paradigm) you can think of it as e.g. a

- **2-dimensional array**, C/C++ :: double A[3][3]; /* A is a 3-by-3 matrix */
- C/C++ :: A[0][0] = 1; /* Assigning 1 to \(a_{11}\) */
- C/C++ :: A[2][2] = 14; /* Assigning 14 to \(a_{33}\) */
- C/C++ :: Yes, some languages count from 0 to (n-1); others from 1 to n.

- or a **container class.**

Types of Matrices

- When \(A \in \mathbb{R}^{n \times n}\), i.e. the matrix has the same number of rows and columns, it is a **square matrix**

- A matrix is **diagonal** if all entries \(a_{ij} = 0\) for all \(i \neq j\). (Only entries of the type \(a_{ii}\) are non-zero.

- A square matrix is **upper triangular** if all entries \(a_{ij} = 0\) for all \(i > j\).
Types of Matrices

- A square matrix is **strictly upper triangular** if all entries $a_{ij} = 0$ for all $i \geq j$.

- A square matrix is **lower triangular** if all entries $a_{ij} = 0$ for all $i < j$.

- A square matrix is **strictly lower triangular** if all entries $a_{ij} = 0$ for all $i \leq j$.

Types of Matrices

- A matrix where all entries are zero is (surprisingly?) called a **zero matrix**.

- A square matrix where all diagonal entries are ones, and the off-diagonal entries are zeros

\[
I_n = \begin{bmatrix}
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
0 & 0 & \cdots & 0 & 1
\end{bmatrix}
\]

is called an **identity matrix**. “\cdots” and “\vdots” denote padding with 0-entries, and “\ddots” diagonal 1-entries; filling the matrix out to its full size (whatever that may be).

Matrices of size $n \times 1$ and $1 \times n$ ⇒ “Vectors”

- A “matrix” with only one column is called a **column vector**:

\[
\vec{v} = \begin{bmatrix}
v_1 \\
v_2 \\
\vdots \\
v_n
\end{bmatrix}
\]

- A “matrix” with only one row is called a **row vector**:

\[
\vec{w}^T = \begin{bmatrix}
w_1 & w_2 & \cdots & w_n
\end{bmatrix}
\]

By mathematical convention a **vector** is a **column vector**, so $\vec{v}$ and $\vec{w}$ are (column) vectors. The notation $\vec{w}^T$ is the **transpose** of the vector $\vec{w}$ into a **row vector**.

Vector “Language”

- The entries of a vector are called its **components**: $v_k$ is the $k^{th}$ component of the vector $\vec{v}$.

- The set (collection) of all (column) vectors with $n$ components is denoted by $\mathbb{R}^n$; we refer to $\mathbb{R}^n$ as a vector space.

It is easy to visualize vectors in $\mathbb{R}^2$; we can think of the vector $\vec{v}$ as an arrow from the origin $(0, 0)$ to the point $(x, y) = (v_1, v_2)$:

In the figure we have

\[
\vec{u} = \begin{bmatrix}
1 \\
2
\end{bmatrix}, \quad \vec{v} = \begin{bmatrix}
1 \\
3
\end{bmatrix}, \quad \vec{w} = \begin{bmatrix}
2 \\
2
\end{bmatrix}
\]

Without confusion, we can just let the terminal points $(x, y) = (v_1, v_2)$ represent the vectors.
Adding Vectors

With:
\[ \vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \]

We can graphically show how to add vectors:

That is
\[ \vec{u} + \vec{v} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad \vec{u} + \vec{v} + \vec{w} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}. \]

Making Use of our Matrix – Vector Notation

Now, given a linear system:
\[
\begin{align*}
2x &+ 8y + 4z = 2 \\
2x &+ 5y + z = 5 \\
4x &+ 10y - z = 1
\end{align*}
\]

We can extract the **Coefficient Matrix** (containing the coefficients of the unknown variables in the system)
\[
\begin{bmatrix}
2 & 8 & 4 \\
2 & 5 & 1 \\
4 & 10 & -1
\end{bmatrix},
\]
or the **augmented matrix**
\[
\begin{bmatrix}
2 & 8 & 4 & 2 \\
2 & 5 & 1 & 5 \\
4 & 10 & -1 & 1
\end{bmatrix},
\]

which captures all the information in the linear system.

The Augmented Matrix

Often, we separate the coefficients from the right-hand-side information in the Augmented Matrix:
\[
\begin{bmatrix}
2 & 8 & 4 & 2 \\
2 & 5 & 1 & 5 \\
4 & 10 & -1 & 1
\end{bmatrix} \sim \begin{bmatrix}
2 & 8 & 4 & 2 \\
2 & 5 & 1 & 5 \\
4 & 10 & -1 & 1
\end{bmatrix}
\]

Solving Linear Systems Using the Augmented Matrix

We can solve the linear system by manipulating the Augmented Matrix:
\[
\begin{bmatrix}
2 & 8 & 4 & 2 \\
2 & 5 & 1 & 5 \\
4 & 10 & -1 & 1
\end{bmatrix} / 2
\]
\[
\begin{bmatrix}
1 & 4 & 2 & 1 \\
2 & 5 & 1 & 5 \\
4 & 10 & -1 & 1
\end{bmatrix} -2r_1
\]
\[
\begin{bmatrix}
1 & 4 & 2 & 1 \\
0 & -3 & -3 & 3 \\
0 & -6 & -9 & -3
\end{bmatrix} / (-3)
\]
The leading variable in each equation does not appear in any other equation.

The (Math) World is not limited to 2, or 3 Variables (Dimensions)

We can (easily?) imagine a system of 4 linear equations with 7 unknowns:

\[
\begin{align*}
    x_1 &- x_2 + 4x_5 = 77777x_7 = 1 \\
    x_3 &- 2x_5 - 777x_7 = 2 \\
    x_4 &- 3 - 77x_7 = 3 \\
    x_6 &- 4 + 7x_7 = 4
\end{align*}
\]

so that solving for the leading variables\(^*\) gives:

\[
\begin{align*}
    x_1 &= 1 + x_2 - 4x_5 - 7777x_7 \\
    x_3 &= 2 - 2x_5 + 777x_7 \\
    x_4 &= 3 - 77x_7 \\
    x_6 &= 4 + 7x_7
\end{align*}
\]

* Leading variables are the first ones to appear in each equation (after elimination); here \(x_1, x_3, x_4,\) and \(x_6.\)

What Makes a System “Easy” to Solve?

Three properties make a system “easy” to solve:

P1 The leading coefficient in each equation is 1.

P2 The leading variable in each equation does not appear in any other equation.

P3 The leading variables appear in “natural order,” with increasing indices as we go down the system: \(x_1, x_3, x_4,\) and \(x_6\) as opposed to any other ordering.

If/when the system does not satisfy these properties, we use elimination to get there...
Another Example... Keeping P1, P2, P3 in mind...

We go straight to the Augmented Matrix:

\[
\begin{bmatrix}
2 & 4 & -2 & 2 & 4 \\
1 & 2 & -1 & 2 & 0 \\
3 & 6 & -2 & 1 & 9 \\
5 & 10 & -4 & 5 & 9
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & -1 & 1 & 2 \\
1 & 2 & -1 & 2 & 0 \\
3 & 6 & -2 & 1 & 9 \\
5 & 10 & -4 & 5 & 9
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & -1 & 1 & 2 \\
0 & 0 & 0 & 1 & -2 \\
0 & 1 & -2 & 3 & -2 \\
0 & 0 & 1 & 0 & -1
\end{bmatrix}
\]

... Identifying the Solutions

\[
\begin{bmatrix}
x_1 & x_2 & x_3 & x_4 & x_5 \\
1 & 2 & 3 & 4 & 2
\end{bmatrix}
\]

So that:

\[
x_1 = 2 - 2x_2 - 3x_5 \\
x_3 = 4 + x_5 \\
x_4 = 3 + 2x_5
\]

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{bmatrix} = \begin{bmatrix}
2 - 2s - 3t \\
s \\
4 + s \\
3 + 2t \\
t
\end{bmatrix}
\]

Parameters: \( x_2 = s, \ x_5 = t \)

Solving a System of Linear Equations

"Elimination Strategy"

We go equation-by-equation from top to bottom (\( i \) runs from 1 to \( n \)):

- For the \( i^{th} \) equation: if the leading variable is \( x_j \) with non-zero coefficient \( c \); divide the equation by \( c \) to make the leading coefficient 1.
- Eliminate \( x_j \) from all other equations.
- Go to the next equation.

"Exit Strategy"

- If we get \( s = 0 \) at any point; then there are no solutions. [STOP]
- If we complete without inconsistencies:
  - \textit{rearrange} the equations so that the leading variables are in \textit{natural order}.
  - Solve each equation for the leading variable.
  - Choose parameters for the non-leading variables [if there are any] (appropriate “alphabet soup”).
  - Express leading variables using parameters.

\[
\begin{bmatrix}
1 & 2 & -1 & 0 & 4 \\
0 & 0 & 1 & -2 & 3 \\
0 & 0 & 1 & 0 & -1 \\
0 & 0 & 1 & 0 & -1
\end{bmatrix}
\]

\[+r_3\]

\[
\begin{bmatrix}
1 & 2 & 0 & 0 & 3 \\
0 & 0 & 1 & -2 & 3 \\
0 & 0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

\[\sim r_3\]

\[\sim r_2\]

\[
\begin{bmatrix}
1 & 2 & 0 & 0 & 3 \\
0 & 1 & 0 & 1 & 4 \\
0 & 0 & 1 & 0 & -1
\end{bmatrix}
\]

Leading zeros suppressed for clarity
Reduced Row Echelon Form

After elimination according to this strategy, the matrix is in:

### Reduced Row Echelon Form

A Matrix is said to be in **Reduced Row Echelon Form** if it satisfies the following conditions:

1. If a row has non-zero entries, then the first non-zero entry is a 1, called the **leading 1** (or pivot) of this row.
2. If a column contains a leading 1, then all other entries in that column are 0. [Elimination is Complete]
3. If a row contains a leading 1, then each row above it contains a leading 1 further to the left. [Sorting of Rows]

The last condition implies that rows of 0's, if any, must appear at the bottom of the matrix.

We get to **Reduced Row Echelon Form** by performing

### Elementary Row Operations

- Divide a row by a non-zero scalar
- Subtract a multiple of a row from another row
- Swap two rows

This strategy of solving linear systems by reduction to Reduced Row Echelon Form is referred to as **Gaussian Elimination**, or **Gauss-Jordan Elimination**.

Gauss (1777–1855), Jordan (1842–1899); but the Chinese used it loooong before that.

"Gauss-Jordan Elimination" ↔ RREF
"Gaussian Elimination" ↔ REF (leading variables NOT 1's; LU-factorization)

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### Suggested Problems 1.2

**Available on “Learning Glass” videos:**

1. **(1.2.1)** Find all solutions to a 2-by-3 linear system using elimination.
2. **(1.2.3)** Find all solutions to a 1-by-3 linear system using elimination.
3. **(1.2.9)** Find all solutions to a 3-by-6 linear system using elimination.
4. **(1.2.11)** Find all solutions to a 4-by-4 linear system using elimination.
5. **(1.2.18)** Determine which matrices are in RREF.
6. **(1.2.21)** Find values of matrix entries so that the resulting matrix is in RREF.

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### Lecture – Book Roadmap

<table>
<thead>
<tr>
<th>Lecture</th>
<th>Book, [GS5–]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>§2.2</td>
</tr>
<tr>
<td>1.2</td>
<td>§1.1, §1.3, §2.1, §2.3</td>
</tr>
</tbody>
</table>

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### Metacognitive Exercise — Thinking About Thinking & Learning

<table>
<thead>
<tr>
<th>I know / learned</th>
<th>Almost there</th>
<th>Huh?!?</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Right After Lecture</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>After Thinking / Office Hours / SI-session</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>After Reviewing for Midterm/Final</strong></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

#### Problem Statements 1.2

**Right After Lecture**

(1.2.1) Find all solutions to the linear system using elimination:

\[
\begin{align*}
x + y - 2z &= 5 \\
2x + 3y + 4z &= 2
\end{align*}
\]

(1.2.3) Find all solutions to the linear system using elimination:

\[
\begin{align*}
x + 2y + 3z &= 4
\end{align*}
\]

**After Thinking / Office Hours / SI-session**

(1.2.9), (1.2.11)

(1.2.9) Find all solutions to the linear system using elimination:

\[
\begin{align*}
x_4 + 2x_5 - x_6 &= 2 \\
x_1 + 2x_2 + 2x_3 &= 0 \\
x_1 + 2x_2 - 2x_3 &= 0
\end{align*}
\]

(1.2.11) Find all solutions to the linear system using elimination:

\[
\begin{align*}
x_1 + 2x_3 + 4x_4 &= -8 \\
x_2 - 3x_3 - x_4 &= 6 \\
3x_1 + 4x_2 - 6x_3 + 8x_4 &= 0 \\
- x_2 + 3x_3 + 4x_4 &= -12
\end{align*}
\]

**After Reviewing for Midterm/Final**

(1.2.18)

(1.2.18) Determine which of the matrices are in Reduced Row Echelon Form:

- \[
\begin{bmatrix}
1 & 2 & 0 & 2 & 0 \\
0 & 0 & 1 & 3 & 0 \\
0 & 0 & 1 & 4 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
- \[
\begin{bmatrix}
0 & 1 & 2 & 0 & 3 \\
0 & 0 & 1 & 4 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
- \[
\begin{bmatrix}
1 & 2 & 0 & 3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 2
\end{bmatrix}
\]
- \[
\begin{bmatrix}
0 & 1 & 2 & 3 & 4
\end{bmatrix}
\]
(1.2.21) For which values of \(a, b, c, d,\) and \(e\) is the following matrix in reduced-row-echelon-form?

\[
\begin{bmatrix}
1 & a & b & 3 & 0 & -2 \\
0 & 0 & c & 1 & d & 3 \\
0 & 0 & 0 & 1 & 1 & 1
\end{bmatrix}
\]