1. Student Learning Objectives
   - SLOs: Matrices, Vectors, ...

2. Matrices, Vectors; Gauss-Jordan Elimination
   - Matrix – Vector Notation
   - Back to Solving Linear Systems
   - Summarizing

3. Suggested Problems
   - Suggested Problems
   - Lecture–Book Roadmap
After this lecture you should:

- Know basic language and concepts:
  - Matrices, vectors, and their components
  - Matrix types: square, diagonal, triangular, zero, identity
  - The collection of all \( n \)-vectors, denoted \( \mathbb{R}^n \) is a vector space

- Vector addition

- Know the difference between the Coefficient matrix, and the Augmented matrix; their uses in the solution of linear systems

- Know how to use elimination to identify leading and non-leading (a.k.a. *free variables*), and when necessary introduce parameters to express *all* solutions of linear systems.

- Know what **Reduced–Row–Echelon–Form** (RREF) of a Matrix is, and how to achieve it using elementary row operations.
Our current “business” is manipulating linear systems, of the form

\[
\begin{align*}
3x + 21y - 3z &= 0 \\
-6x - 2y - z &= 62 \\
2x - 3y + 8z &= 32
\end{align*}
\]

into a form which reveals the values of \(x\), \(y\), and \(z\):

\[
\begin{align*}
x &= -\frac{3574}{281} \\
y &= \frac{844}{281} \\
z &= \frac{2334}{281}
\end{align*}
\]

We achieve this by cleverly adding/subtracting rows (equations) from each other.
We realize that all the important information is in the coefficients (numbers), and that the variables \((x, y, z)\) just get carried around. We can “encode” all the information about the linear system

\[
\begin{align*}
3x & + 21y & - 3z & = 0 \\
-6x & - 2y & - z & = 62 \\
2x & - 3y & + 8z & = 32
\end{align*}
\]

in a **matrix**

\[
\begin{bmatrix}
3 & 21 & -3 & 0 \\
-6 & -2 & -1 & 62 \\
2 & -3 & 8 & 32
\end{bmatrix}
\]

or, sometimes:

\[
\begin{bmatrix}
3 & 21 & -3 & 0 \\
-6 & -2 & -1 & 62 \\
2 & -3 & 8 & 32
\end{bmatrix}
\]

**Augmented Matrix**

**Augmented Matrix with Coefficient Matrix** and right-hand-side “separated.”
Row – Column Indexing

Ponder the matrix “A” with 3 rows, and 4 columns:

\[
A = \begin{bmatrix}
3 & 21 & -3 & 0 \\
-6 & -2 & -1 & 62 \\
2 & -3 & 8 & 32 \\
\end{bmatrix} = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & a_{14} \\
a_{21} & a_{22} & a_{23} & a_{24} \\
a_{31} & a_{32} & a_{33} & a_{34} \\
\end{bmatrix}
\]

that is, usually we refer to the entries of a matrix \( A \) (upper-case), using double subscripts \( a_{ij} \) (lower-case); the subscripts \( i \), and \( j \) are “standard” but \( r \) (row) and \( c \) (column) would be more intuitive.

Sometimes you see the notation \( A \in \mathbb{R}^{3\times4} \) to denote a 3-by-4 (always [\text{Rows-by-Columns}]) matrix where the entries are real (\( \mathbb{R} \)) numbers.

Note: The entries can be other mathematical objects, e.g. complex numbers, \( \mathbb{C} \), polynomials, etc... but we will work with \( \mathbb{R} \) for quite while.
Types of Matrices

- When $A \in \mathbb{R}^{n \times n}$, i.e. the matrix has the same number of rows and columns, it is a **square matrix**

- A matrix is **diagonal** if all entries $a_{ij} = 0$ for all $i \neq j$. (Only entries of the type $a_{ii}$ are non-zero.

- A square matrix is **upper triangular** if all entries $a_{ij} = 0$ for all $i > j$. 
Types of Matrices

- A square matrix is **strictly upper triangular** if all entries $a_{ij} = 0$ for all $i \geq j$.

- A square matrix is **lower triangular** if all entries $a_{ij} = 0$ for all $i < j$.

- A square matrix is **strictly lower triangular** if all entries $a_{ij} = 0$ for all $i \leq j$. 
Types of Matrices

- A matrix where all entries are zero is (surprisingly?) called a **zero matrix**.

- A square matrix where all diagonal entries are **ones**, and the off-diagonal entries are **zeros**

\[
\begin{bmatrix}
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
0 & 0 & \cdots & 0 & 1
\end{bmatrix}
\]

is called an **identity matrix**. ("\cdots" and "\cdot" denote padding with 0-entries, and "\cdot \cdot \cdot" diagonal 1-entries; filling the matrix out to its full size (whatever that may be).
Matrices of size $n \times 1$ and $1 \times n \Rightarrow \text{“Vectors”}

- A “matrix” with only one column is called a **column vector**:

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

- A “matrix” with only one row is called a **row vector**:

$$\vec{w}^T = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix}$$

By mathematical convention a **vector** is a **column vector**; so $\vec{v}$ and $\vec{w}$ are (column) vectors. The notation $\vec{w}^T$ is the transpose of the vector $\vec{w}$ into a **row vector**.
The entries of a vector are called its components; \( v_k \) is the \( k^{\text{th}} \) component of the vector \( \vec{v} \).

The set (collection) of all (column) vectors with \( n \) components is denoted by \( \mathbb{R}^n \); we refer to \( \mathbb{R}^n \) as a vector space.

It is easy to visualize vectors in \( \mathbb{R}^2 \); we can think of the vector \( \vec{v} \) as an arrow from the origin \((0, 0)\) to the point \((x, y) = (v_1, v_2)\):

In the figure we have

\[
\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 2 \\ 2 \end{bmatrix},
\]

Without confusion, we can just let the terminal points \((x, y) = (v_1, v_2)\) represent the vectors.
Adding Vectors

With:

\[ \vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \]

We can graphically show how to add vectors:

That is

\[ \vec{u} + \vec{v} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \quad \vec{u} + \vec{v} + \vec{w} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}. \]
Making Use of our Matrix – Vector Notation

Now, given a linear system:

\[
\begin{align*}
2x + 8y + 4z &= 2 \\
2x + 5y + z &= 5 \\
4x + 10y - z &= 1
\end{align*}
\]

We can extract the **Coefficient Matrix** (containing the coefficients of the unknown variables in the system)

\[
\begin{bmatrix}
2 & 8 & 4 \\
2 & 5 & 1 \\
4 & 10 & -1
\end{bmatrix},
\]

or the **augmented matrix**

\[
\begin{bmatrix}
2 & 8 & 4 & 2 \\
2 & 5 & 1 & 5 \\
4 & 10 & -1 & 1
\end{bmatrix},
\]

which captures all the information in the linear system.
Often, we separate the coefficients from the right-hand-side information in the Augmented Matrix:

\[
\begin{bmatrix}
2 & 8 & 4 & 2 \\
2 & 5 & 1 & 5 \\
4 & 10 & -1 & 1
\end{bmatrix}
\sim
\begin{bmatrix}
2 & 8 & 4 & 2 \\
2 & 5 & 1 & 5 \\
4 & 10 & -1 & 1
\end{bmatrix}
\]
Solving Linear Systems Using the Augmented Matrix

We can solve the linear system by manipulating the Augmented Matrix:

\[
\begin{bmatrix}
2 & 8 & 4 & 2 \\
2 & 5 & 1 & 5 \\
4 & 10 & -1 & 1 \\
\end{bmatrix}
/ 2
\]

\[
\begin{bmatrix}
1 & 4 & 2 & 1 \\
2 & 5 & 1 & 5 \\
4 & 10 & -1 & 1 \\
\end{bmatrix}
-2r_1
\]

\[
\begin{bmatrix}
1 & 4 & 2 & 1 \\
0 & -3 & -3 & 3 \\
0 & -6 & -9 & -3 \\
\end{bmatrix}
/ (-3)
\]

Peter Blomgren, ⟨blomgren.peter@gmail.com⟩
Solving Linear Systems Using the Augmented Matrix

\[
\begin{bmatrix}
1 & 4 & 2 & | & 1 \\
0 & 1 & 1 & | & -1 \\
0 & 2 & 3 & | & 1
\end{bmatrix}
\]

\(-2r_2\)

\[
\begin{bmatrix}
1 & 4 & 2 & | & 1 \\
0 & 1 & 1 & | & -1 \\
0 & 0 & 1 & | & 3
\end{bmatrix}
\]

\(-r_3\)

\[
\begin{bmatrix}
1 & 4 & 0 & | & -5 \\
0 & 1 & 0 & | & -4 \\
0 & 0 & 1 & | & 3
\end{bmatrix}
\]

\(-4r_2\)

\[
\begin{bmatrix}
1 & 0 & 0 & | & 11 \\
0 & 1 & 0 & | & -4 \\
0 & 0 & 1 & | & 3
\end{bmatrix}
\]

\[
x \quad y \quad z = \begin{bmatrix}
11 \\
-4 \\
3
\end{bmatrix}
\]
We can (easily?) imagine a system of 4 linear equations with 7 unknowns:

\[
\begin{align*}
    x_1 &- x_2 &+ 4x_5 &\quad 7777x_7 &= 1 \\
    x_3 &+ 2x_5 &- 777x_7 &= 2 \\
    x_4 & & & 77x_7 &= 3 \\
    x_6 &- 7x_7 &= 4
\end{align*}
\]
The (Math) World is not limited to 2, or 3 Variables (Dimensions)

We can (easily?) imagine a system of 4 linear equations with 7 unknowns:

\[
\begin{align*}
\begin{array}{cccc}
  x_1 & - & x_2 & + 4x_5 \\
  x_3 & + & 2x_5 & - 777x_7 \\
  x_4 & & & - 7x_7 \\
  x_6 & & & - 77x_7 \\
\end{array}
\end{align*}
\]

so that solving for the leading variables* gives:

\[
\begin{align*}
  x_1 &= 1 + x_2 - 4x_5 - 777x_7 \\
  x_3 &= 2 - 2x_5 + 777x_7 \\
  x_4 &= 3 - 77x_7 \\
  x_6 &= 4 + 7x_7 \\
\end{align*}
\]

* **Leading variables** are the first ones to appear in each equation (after elimination); here \( x_1, x_3, x_4, \) and \( x_6 \).
Infinitely Many Solutions

If we parameterize (the non-leading, or “free” variables): \( x_2 = s \), \( x_5 = t \), and \( x_7 = u \), we can write the infinitely many solutions:

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5 \\
  x_6 \\
  x_7
\end{bmatrix}
= \begin{bmatrix}
  1 & + & s & - & 4t & - & 7777u \\
  s & 2 & - & 2t & + & 777u \\
  3 & t & - & 77u \\
  4 & + & 7u & u
\end{bmatrix}
= \begin{bmatrix}
  1 & 0 & 1 & -4 & -7777 \\
  0 & 1 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 \\
  0 & 0 & 0 & 1 & 0 \\
  0 & 0 & 0 & 0 & 1 \\
  0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  1 \\
  2 \\
  3 \\
  4 \\
  s \\
  t \\
  u
\end{bmatrix}
\]

If we plug that into the original system of linear equations we see that it indeed is the (collection of) solution(s)!

Note that \( s \), \( t \), and \( u \) are allowed to take any values in \( \mathbb{R} \) (independent of each other)...

Peter Blomgren, \langle blomgren.peter@gmail.com \rangle

Lecture Notes #1.2 — Matrices, Vectors, ... — (18/27)
What Makes a System “Easy” to Solve?

Three properties make a system “easy” to solve:

**P1** The leading coefficient in each equation is 1.

**P2** The leading variable in each equation does not appear in any other equation.

**P3** The leading variables appear in “natural order,” with increasing indices as we go down the system: $x_1$, $x_3$, $x_4$, and $x_6$ as opposed to any other ordering.

If/when the system does not satisfy these properties, we use elimination to get there...
Another Example...

We go straight to the Augmented Matrix:

\[
\begin{bmatrix}
2 & 4 & -2 & 2 & 4 & 2 \\
1 & 2 & -1 & 2 & 0 & 4 \\
3 & 6 & -2 & 1 & 9 & 1 \\
5 & 10 & -4 & 5 & 9 & 9 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & -1 & 1 & 2 & 1 \\
1 & 2 & -1 & 2 & 0 & 4 \\
3 & 6 & -2 & 1 & 9 & 1 \\
5 & 10 & -4 & 5 & 9 & 9 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & -1 & 1 & 2 & 1 \\
0 & 0 & 0 & 1 & -2 & 3 \\
0 & 0 & 1 & -2 & 3 & -2 \\
0 & 0 & 1 & 0 & -1 & 4 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & -1 & 1 & 2 & 1 \\
0 & 0 & 0 & 1 & -2 & 3 \\
0 & 0 & 1 & -2 & 3 & -2 \\
0 & 0 & 1 & 0 & -1 & 4 \\
\end{bmatrix}
\]
Another Example...

Keeping P1, P2, P3 in mind...

\[
\begin{bmatrix}
1 & 2 & -1 & 0 & 4 & -2 \\
0 & 0 & 0 & 1 & -2 & 3 \\
0 & 0 & 1 & 0 & -1 & 4 \\
0 & 0 & 1 & 0 & -1 & 4 \\
\end{bmatrix} + r_3
\]

\[
\begin{bmatrix}
1 & 2 & 0 & 0 & 3 & 2 \\
0 & 0 & 0 & 1 & -2 & 3 \\
0 & 0 & 1 & 0 & -1 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} - r_3
\]

\[
\begin{bmatrix}
1 & 2 & 0 & 0 & 3 & 2 \\
0 & 0 & 0 & 1 & -2 & 3 \\
0 & 0 & 1 & 0 & -1 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \rightarrow r_3
\]

\[
\begin{bmatrix}
1 & 2 & 0 & 0 & 3 & 2 \\
1 & 0 & -1 & 4 \\
1 & -2 & 3 \\
\end{bmatrix}
\]

leading zeros

suppressed

for clarity
... Identifying the Solutions

\[
\begin{bmatrix}
  x_1 & x_2 & x_3 & x_4 & x_5 \\
  1 & 2 & 3 & 4 & 5 \\
  1 & -1 & 1 & -2 & 3 \\
\end{bmatrix}
\]

So that:

\[
\begin{align*}
x_1 &= 2 - 2x_2 - 3x_5 \\
x_3 &= 4 + x_5 \\
x_4 &= 3 + 2x_5
\end{align*}
\]

or

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4 \\
  x_5
\end{bmatrix}
= \begin{bmatrix}
  2 - 2s - 3t \\
  s \\
  4 + t \\
  3 + 2t \\
  t
\end{bmatrix}
\]

Peter Blomgren, \{blomgren.peter@gmail.com\}
Solving a System of Linear Equations

“Elimination Strategy”

We go equation-by-equation from top to bottom ($i$ runs from 1 to $n$):

- For the $i^{th}$ equation: if the leading variable is $x_j$ with non-zero coefficient $c$; divide the equation by $c$ to make the leading coefficient 1.
- Eliminate $x_j$ from all other equations.
- Go to the next equation.

“Exit Strategy”

- If we get zero = nonzero at any point; then there are no solutions. [STOP]
- If we complete without inconsistencies:
  - **rearrange** the equations so that the leading variables are in “natural order”
  - Solve each equation for the leading variable
  - Choose parameters for the **non-leading variables** [if there are any] (appropriate “alphabet soup”)
  - Express leading variables using parameters.
After elimination according to this strategy, the matrix is in:

**Reduced Row Echelon Form**

A Matrix is said to be in \textit{Reduced Row Echelon Form} if it satisfies the following conditions:

1. If a row has non-zero entries, then the first non-zero entry is a 1, called \textit{the leading 1} (or \textit{pivot}) of this row.

2. If a column contains a leading 1, then all other entries in that column are 0. [\textit{Elimination is Complete}]

3. If a row contains a leading 1, then each row above it contains a leading 1 further to the left. [\textit{Sorting of Rows}]

The last condition implies that rows of 0’s, if any, must appear at the bottom of the matrix.
We get to *Reduced Row Echelon Form* by performing

**Elementary Row Operations**
- Divide a row by a non-zero scalar
- Subtract a multiple of a row from another row
- Swap two rows

This strategy of solving linear systems by reduction to Reduced Row Echelon Form is referred to as *Gaussian Elimination*, or *Gauss-Jordan Elimination*.

Gauss (1777–1855), Jordan (1842–1899); but the Chinese used it loooong before that.

“Gauss-Jordan Elimination” $\rightsquigarrow$ RREF
“Gaussian Elimination” $\rightsquigarrow$ REF (leading variables NOT 1's; $LU$-factorization)
Available on Learning Glass videos:
1.2 — 1, 3, 9, 11, 18, 21

Not (yet?) on Learning Glass:
GS5 — $n.m$: $x$, $y$, $z$
<table>
<thead>
<tr>
<th>Lecture</th>
<th>Book, [GS5–]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>§2.2</td>
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<tr>
<td>1.2</td>
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