Student Learning Objectives
SLOs: Solutions of Linear systems; Matrix Algebra

Know what the **Rank** of a Matrix is; and its connection to total/leading/free variables and the number of solutions of a linear system

Know the Fundamentals of:
- Matrix-Vector algebra
- Vector-Vector Dot Product / Inner Product
- Matrix-Vector Product: Linear combinations

How Many Solutions Are There?!?

That's a good question; and it ties in with last lecture...

Let's ponder the three (eliminated) systems:

\[
\begin{pmatrix}
1 & 2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\quad \begin{pmatrix}
1 & 2 & 0 & 1 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix}
\quad \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 0 & 3 \\
\end{pmatrix}
\]

Where the leading coefficients have been circled in red. Notice that a. we did not circle the 1 in the third row, since it belongs to the right-hand-side (and NOT the coefficient matrix).
System a. — No Solutions

\[
\begin{bmatrix}
1 & 2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Here, the third row shows that there are no solutions to this system. We say that the system is inconsistent.

System b. — Infinitely Many Solutions

\[
\begin{bmatrix}
1 & 2 & 0 & 1 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

We are left with one un-determined (free) variable; and introduce a parameter (let’s pick the Greek letter \( \eta \) for fun), and write the infinitely many solutions as:

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
1 - 2\eta \\
\eta \\
2
\end{bmatrix} + \begin{bmatrix}
-2 \\
1 \\
0
\end{bmatrix}, \quad \text{where } \eta \in \mathbb{R} \Rightarrow \eta \in [−\infty, +\infty]
\]

System c. — One (Unique) Solutions

\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 3
\end{bmatrix}
\]

Here, there are no un-determined (free) variables; so there’s only one solution.

Theorem (Number of Solutions of a Linear System)

A system of equations is said to be consistent if there is at least one solution; it is inconsistent if there are no solutions.

A linear system is inconsistent if and only if the reduced row-echelon form of its augmented matrix contains the row

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 1
\end{bmatrix},
\]

representing the equation “0 = 1.”

If a linear system is consistent, then it has either

- infinitely many solutions, if there is at least one free variable, or
- exactly one solution, if all the variables are leading.
The Rank of a Matrix

Definition (The RANK of a Matrix)

The rank of a matrix is the number of leading 1s in $\text{rref}(A)$ — the Reduced Row Echelon Form of $A$ — and is denoted $\text{rank}(A)$.

Definition (Full RANK)

If $A \in \mathbb{R}^{n \times n}$ (a square matrix of size $n$), and $\text{rank}(A) = n$, then the matrix is said to have full rank.

Heads-up! In terms of linear systems, the important rank is that of the coefficient matrix...

Properties of the rank($A$)

Property #1a, and #1b

The inequalities

$$\text{rank}(A) \leq n, \quad \text{and} \quad \text{rank}(A) \leq m$$

hold.

“Proof:” If we transform $A$ into $\text{rref}(A)$, there is at most one leading 1 in each of the $n$ rows (showing #1a); and there is at most one leading 1 in each of the $m$ columns (showing #1b).

Property #2

If the system is inconsistent, then

$$\text{rank}(A) < n.$$ 

“Proof:” For an inconsistent matrix $A$, $\text{rref}(A)$ will contain (at least) a row of the form $[0 \ 0 \ 0 \ 0 \ | \ 1]$, so the rank can be at most $n - 1$.

Property #3

If the system has exactly one solution, then

$$\text{rank}(A) = m.$$ 

“Proof:” A leading 1 for each variable leaves no free (un-determined) variables.

Property #4

If the system has infinitely many solutions, then

$$\text{rank}(A) < m.$$ 

“Proof:” In this case, there’s at least one free (un-determined) variable, which does not have a corresponding leading 1.
Properties of the \( \text{rank}(A) \)

It is true that (for \( A \in \mathbb{R}^{n \times m} \))

\[
\text{\# Free Variables} = \text{\# Total Variables} - \text{\# Leading Variables} = m - \text{rank}(A).
\]

More Mathematical Language: The Contrapositive

**Definition (The Contrapositive of a Statement)**

The contrapositive of a logic statement “if \( p \) then \( q \)”, in math notation: \( p \rightarrow q \); is: “if not-\( q \) then not-\( p \)”, notation: \( \sim q \rightarrow \sim p \).

The contrapositive of

\[
\text{if you are in this room} \quad \text{then you are in this building}, \quad p \rightarrow q
\]

is

\[
\text{if you are not in this building} \quad \text{then you are not in this room}, \quad \sim q \rightarrow \sim p
\]

A statement and its contrapositive are logically equivalent; that is, if the statement is true, then the contrapositive is true.

Using the Contrapositive

We have some true statements (for \( A \in \mathbb{R}^{n \times m} \)):

- if the system is inconsistent, then \( \text{rank}(A) < n \).
- if the system has exactly one solution, then \( \text{rank}(A) = m \).
- if the system has infinitely many solutions, then \( \text{rank}(A) < m \).

Using the contrapositive, we immediately can say that

- if \( \text{rank}(A) = n \), then the system is consistent.
- if \( \text{rank}(A) < m \), then the system has either no solutions, or infinitely many solutions.
- if \( \text{rank}(A) = m \), then the system has no solutions, or exactly one solution.

The number of equations vs. the number of unknowns

**Theorem (#Equations vs. #Unknowns)**

- **statement**: If a linear system has exactly one solution, then there must be at least as many equations as there are variables; \( m \leq n \) using previous notation. [The coefficient matrix is either square, or “tall and skinny.”]
- **contrapositive**: If a linear system has fewer equations than unknowns \( (n < m) \), then it either has no solutions or infinitely many solutions. [The coefficient matrix is “short and wide.”]

**Proof (of statement).**

A system with exactly one solution has \( m = \text{rank}(A) \) [PROPERTY \#3]; further \( \text{rank}(A) \leq n \) [PROPERTY \#1A], therefore

\[
m = \text{rank}(A) \leq n
\]

which shows \( m \leq n \).
The rank of the coefficient matrix $A$ satisfies

A linear system of $n$ equations (rows in the coefficient matrix) in $n$ variables (columns in the coefficient matrix) has a unique solution if and only if the rank of the coefficient matrix $A$ satisfies $\text{rank}(A) = n$. When that is true the Reduced Row Echelon Form of $A$ satisfies

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

that is $\text{rref}(A)$ is the $n \times n$ identity matrix, usually denoted $I_n$.

Theorem (Systems of $n$ equations in $n$ variables)

We now define ways that our Matrix and Vector objects can “interact;” we are adding some “verbs” to our Mathematical language!

Definition (Matrix Sums)

The sum of two matrices of the same size $A, B \in \mathbb{R}^{n \times m}$ is determined by the entry-by-entry sums, that is if

$$C = A + B$$

then $C \in \mathbb{R}^{n \times m}$, and $c_{ij} = a_{ij} + b_{ij}$ for $i \in [1, \ldots, n], j \in [1, \ldots, m]$.

Definition (Scalar Multiple of a Matrix)

If $A \in \mathbb{R}^{n \times m}$ is a matrix, and $\rho \in \mathbb{R}$ is a real scalar, then the scalar-matrix-product

$$C = \rho A$$

gives $C \in \mathbb{R}^{n \times m}$, and $c_{ij} = \rho a_{ij}$.

Definition (Matrix-Vector product)

If $A \in \mathbb{R}^{n \times m}$ matrix with row-vectors $\mathbf{r}_1^T, \ldots, \mathbf{r}_n^T \in \mathbb{R}^m$, and $\mathbf{x} \in \mathbb{R}^m$ is a (column) vector, then

$$A\mathbf{x} = \begin{bmatrix} \mathbf{r}_1^T \\ \vdots \\ \mathbf{r}_n^T \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{r}_1^T \mathbf{x} \\ \vdots \\ \mathbf{r}_n^T \mathbf{x} \end{bmatrix} \equiv \begin{bmatrix} \mathbf{r}_1^T \cdot \mathbf{x} \\ \vdots \\ \mathbf{r}_n^T \cdot \mathbf{x} \end{bmatrix}$$

Using Inner Product Notation Using Dot Product Notation

The $i^{\text{th}}$ component of the resulting vector $\mathbf{y} = A\mathbf{x}$ is given by the dot (inner) product of the $i^{\text{th}}$ row of $A$ and the vector $\mathbf{x}$. Note that if $m \neq n$ then $\mathbf{y} \in \mathbb{R}^n$ is not the same size as $\mathbf{x} \in \mathbb{R}^m$.

Note: The way we have defined the dot product it is not row/column sensitive. However if you stick with the standard notation that “vectors” are column-vectors, it common to see the equivalent notation:

$$\mathbf{v}^T \mathbf{w} \equiv \mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + \ldots + v_n w_n = \sum_{k=1}^n v_k w_k,$$

which tends to be referred to as the inner product.
Matrix Algebra — (21/26)

The Number of Solutions to a System of Linear Equations
Definitions and Rules of Matrix Algebra
Suggested Problems

Size and Shape Do Matter in Matrix-Vector Multiplication

For the matrix-vector product to make sense, the matrix $A \in \mathbb{R}^{n \times m}$ and the vector $\vec{x} \in \mathbb{R}^m \equiv \mathbb{R}^{m \times 1}$ must have compatible sizes:

$$
\begin{bmatrix}
A & \vec{x} \\
\end{bmatrix} = \begin{bmatrix}
\vec{y}
\end{bmatrix}
$$

Looking Ahead (Matrix Multiplication): thinking about size, it’s probably OK to multiply $A \in \mathbb{R}^{n \times m}$ and $B \in \mathbb{R}^{m \times p}$, a solid “guess” for the size of the result? —

$$
\begin{bmatrix}
A & B
\end{bmatrix} = \begin{bmatrix}
C
\end{bmatrix}
$$

however the product $BA$ does not make sense (unless $n = p$).

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Thinking about $A\vec{x}$ in a different way

So far, we have thought of the components of $A\vec{x}$ as the result of dot-products of the rows of $A$ and the vector $\vec{x}$; to inspire a different view:

Consider $A \in \mathbb{R}^{2 \times 3}$ and $\vec{x} \in \mathbb{R}^3$, then

$$
A\vec{x} = \begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = x_1 a_{11} + x_2 a_{12} + x_3 a_{13}
$$

We realize that

$$
\begin{bmatrix}
a_{11} x_1 + a_{12} x_2 + a_{13} x_3 \\
a_{21} x_1 + a_{22} x_2 + a_{23} x_3
\end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}
$$

Which means that we can think of $\vec{y} = A\vec{x}$ as a sum of vectors (where the vectors are the columns of $A$, scaled by the components of $\vec{x}$)

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Thinking about $A\vec{x}$ as the Linear Combination of the Columns

Theorem (The Product $A\vec{x}$ in Terms of the Columns of $A$)

If the column vectors of an $n \times m$ matrix $A$ are $\vec{v}_1, \ldots, \vec{v}_m$ and $\vec{x} \in \mathbb{R}^m$ with components $x_1, \ldots, x_m$, then

$$
A\vec{x} = \begin{bmatrix}
\vec{v}_1 & \ldots & \vec{v}_m
\end{bmatrix}
\begin{bmatrix}
x_1 \\
\vdots \\
x_m
\end{bmatrix} = x_1 \vec{v}_1 + \cdots + x_m \vec{v}_m.
$$

Definition (Linear Combinations)

A vector $\vec{b}$ in $\mathbb{R}^n$ is called a linear combination of the vectors $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$ if there exists scalars $x_1, \ldots, x_m$ such that

$$
\vec{b} = x_1 \vec{v}_1 + \cdots + x_m \vec{v}_m.
$$

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Available on Learning Glass videos:
1.3 — 1, 2, 3, 7, 13, 22, 23, 27, 37, 46, 55

Not (yet?) on Learning Glass:
GS5 — $n \cdot m$: $x, y, z$