Matrix Algebra — (1/37)
Outline

1. Student Learning Objectives
   - SLOs: Solutions of Linear systems; Matrix Algebra

2. The Number of Solutions to a System of Linear Equations
   - Collecting the Results... and Adding More Language / Notation
   - Mathematical Language: Logic
   - Using Logic to Derive More Results Re: Variables and Rank

3. Definitions and Rules of Matrix Algebra
   - Fundamentals of Matrix and Vector Algebra

4. Suggested Problems
   - Suggested Problems 1.3
   - Lecture–Book Roadmap

5. Supplemental Material
   - Metacognitive Reflection
   - Problem Statements 1.3
After this lecture you should:

- Know what the **Rank** of a Matrix is; and its connection to total/leading/free variables and the number of solutions of a linear system
- Know the Fundamentals of:
  - Matrix-Vector algebra
  - Vector-Vector Dot Product / Inner Product
  - Matrix-Vector Product: Linear combinations
How Many Solutions Are There?!

That’s a good question; and it ties in with last lecture...

Let’s ponder the three (augmented, eliminated) systems:

\[
\begin{bmatrix}
1 & 2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 0 & 1 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 3 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Where the leading coefficients have been circled in red. Notice that in a, we did not circle the 1 in the third row, since it belongs to the right-hand-side (and NOT the coefficient matrix).
System a. — No Solutions

\[
\begin{bmatrix}
1 & 2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\text{interpretation}
\begin{align*}
x_1 &= 0 - 2x_2 \\
x_3 &= 0 \\
0 &= 1 \\
0 &= 0 \\
\end{align*}
\]

Here, the third row shows that there are no solutions to this system. We say that the system is inconsistent.
System b. — Infinitely Many Solutions

We are left with one un-determined (free) variable; and introduce a parameter for $x_2$ (let’s pick the Greek letter $\eta$ for fun), and write the infinitely many solutions as:

$$
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix} 1 - 2\eta \\ \eta \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + \eta \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad \text{where } \eta \in \mathbb{R} \iff \eta \in [-\infty, +\infty]
$$
System c. — One (Unique) Solutions

\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 3 \\
\end{bmatrix}
\quad \text{interpretation}
\quad \begin{align*}
x_1 &= 1 \\
x_2 &= 2 \\
x_3 &= 3 \\
\end{align*}
\]

Here, there are no un-determined (free) variables; so there’s only one solution.
Theorem (Number of Solutions of a Linear System)

A system of equations is said to be **consistent** if there is at least one solution; it is **inconsistent** if there are no solutions.

A linear system is inconsistent if and only if the reduced row-echelon form of its augmented matrix contains the row

\[
\begin{bmatrix} 0 & 0 & 0 & 0 & 0 | 1 \end{bmatrix},
\]

representing the equation “\(0 = 1\).”

*If a linear system is consistent, then it has either*

- infinitely many solutions, if there is at least one free variable,
- or
- exactly one solution, if all the variables are leading.*
Definition (The RANK of a Matrix)

The **rank** of a matrix is the number of leading 1s in $\text{rref}(A)$ — the Reduced Row Echelon Form of $A$ — and is denoted $\text{rank}(A)$.

Definition (Full RANK)

If $A \in \mathbb{R}^{n \times n}$ (a square matrix of size $n$), and $\text{rank}(A) = n$, then the matrix is said to have **full rank**.

Heads-up! In terms of linear systems; the important rank is that of the *coefficient matrix*...
Properties of the \texttt{rank}(A)

Consider a matrix \( A \in \mathbb{R}^{n \times m} \), corresponding to a linear system of \( n \) equations with \( m \) unknowns:

\begin{itemize}
  \item \textbf{Property \#1a, and \#1b}
  \item The inequalities
    \[
    \text{rank}(A) \leq n, \quad \text{and} \quad \text{rank}(A) \leq m
    \]
    hold.
\end{itemize}

\textit{“Proof:”} If we transform \( A \) into \texttt{rref}(A), there is \textit{at most} one leading 1 in each of the \( n \) rows (showing \#1a); and there is at most one leading 1 in each of the \( m \) columns (showing \#1b).
Properties of the \text{rank}(A)

Property #2
If the system is inconsistent, then

\[
\text{rank}(A) < n.
\]

\text{“Proof:”} For an inconsistent matrix \( A \), \text{rref}(A) will contain (at least) a row of the form \([ 0 \ 0 \ 0 \ 0 \mid 1 ]\) — which does not have a leading one — so the rank can be at most \((n - 1)\).
Properties of the $\text{rank}(A)$

Property #3
If the system has exactly one solution, then

$$\text{rank}(A) = m.$$  

“Proof:” A leading 1 for each variable leaves no free (un-determined) variables.

Property #4
If the system has infinitely many solutions, then

$$\text{rank}(A) < m.$$

“Proof:” In this case, there’s at least one free (un-determined) variable, which does not have a corresponding leading 1.
Properties of the \( \text{rank}(A) \)

It is true that (for \( A \in \mathbb{R}^{n \times m} \))

\[
\#\text{Free\_Variables} = \#\text{Total\_Variables} - \#\text{Leading\_Variables} = m - \text{rank}(A).
\]
More Mathematical Language: The Contrapositive

Definition (The Contrapositive of a Statement)

The contrapositive of a logic statement “if $p$ then $q$”, in math notation: $p \rightarrow q$; is: “if not-$q$ then not-$p$”, notation: $(\sim q) \rightarrow (\sim p)$.

The contrapositive of

\[
\begin{align*}
\text{if } & \text{you are in this room } \text{then } \text{you are in this building} \\
p & \text{ } \\
\text{is } \\
\text{if } & \text{you are not in this building } \text{then } \text{you are not in this room} \\
(\sim q) & \text{ } (\sim p)
\end{align*}
\]

A statement and its contrapositive are logically equivalent; that is if the statement is true, then the contrapositive is true.
We have some true statements (for $A \in \mathbb{R}^{n \times m}$):

(i) if the system is inconsistent, then $\text{rank}(A) < n$.

(ii) if the system has exactly one solution, then $\text{rank}(A) = m$.

(iii) if the system has infinitely many solutions, then $\text{rank}(A) < m$.

Using the contrapositive, we immediately can say that

(i)* if $\text{rank}(A) = n$, then the system is consistent.

(ii)* if $\text{rank}(A) < m$, then the system has either no solutions, or infinitely many solutions.

(iii)* if $\text{rank}(A) = m$, then the system has no solutions, or exactly one solution.
In all cases below, $A \in \mathbb{R}^{n \times m}$, $\text{rank}(A) \leq \min(n, m)$.

(i) For an \textbf{inconsistent} system, there must be (as least) one row with zeros on one the coefficient-side, and a non-zero on the right-hand-side:

at most \[ \begin{bmatrix} \times & \cdots & \times & \times \\ \vdots & \vdots & \vdots & \vdots \\ \times & \cdots & \times & \times \end{bmatrix} \]

$n - 1$ leading ones

No leading one in this row \[ \begin{bmatrix} \times & \cdots & \times & \times \\ 0 & \cdots & 0 & 1 \end{bmatrix} \]

therefore, $\text{rank}(A) < n$. 
(ii) When a system has exactly one solution, then \( \text{rref}(A) \) must have a \textit{leading one} in each column (no free variables can remain). The number of columns (\( m \)) equals the number of variables; so we must have \( \text{rank}(A) = m \). Note that therefore \( n \geq m \) — there can only be a single \textit{leading one} in each row. We get two cases:

- \((n = m) \Rightarrow \text{rref}(A) = I_n\)
- \((n > m) \Rightarrow \text{Rows (} m + 1 \text{) to (} n \text{)} must be all zeros, with zero right-hand-side.\)

(iii) When a system has infinitely many solutions, there is at least one \textit{free variable}. Therefore \( \text{rref}(A) \) must have at least one column \textit{without} a leading one, which means that \( \text{rank}(A) \leq (m - 1) \). \( \Rightarrow \text{rank}(A) < m \).
Thinking about the contrapositive statements...

(i)* When $\text{rank}(A) = n$, there are leading ones in each row of the reduced system. Therefore, there cannot be any row of the form

$$\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}$$

which would indicate inconsistency. Hence, the system must be consistent. Again, we have two cases:

- $(m = n) \Rightarrow \text{rref}(A) = I_n \Rightarrow$ the solution is unique.
- $(m > n) \Rightarrow$ there are $(m - n)$ free variables $\Rightarrow$ there are infinitely many solutions.
(ii)* When \( \text{rank}(A) < m \), there is at least one column without a leading one \( \Rightarrow \) there is at least one free variable. Note that this does not rule out rows of the form
\[
\begin{bmatrix}
0 & \cdots & 0 & 1
\end{bmatrix}.
\]
if such a row exists, the system is inconsistent and has no solutions, otherwise the system is consistent with (at least) one free variable, and has infinitely many solutions.

(iii)* When \( \text{rank}(A) = m \), there is a leading one in each column \( \Rightarrow \) there are no free variables. If there is a row of the form
\[
\begin{bmatrix}
0 & \cdots & 0 & 1
\end{bmatrix}.
\]
the system is inconsistent and has no solutions, otherwise the system is consistent with a unique solution.
The number of equations vs. the number of unknowns

**Theorem (#Equations vs. #Unknowns)**

- **statement**: If a linear system has exactly one solution, then there must be at least as many equations as there are variables; \( m \leq n \) using previous notation. [The coefficient matrix is either square, or “tall and skinny.”]

- **contrapositive**: If a linear system has fewer equations than unknowns \( n < m \), then it either has no solutions or infinitely many solutions. [The coefficient matrix is “short and wide.”]

**Proof (of statement).**

A system with exactly one solution has \( m = \text{rank}(A) \) [PROPERTY #3]; further \( \text{rank}(A) \leq n \) [PROPERTY #1A], therefore

\[
m = \text{rank}(A) \leq n
\]

which shows \( m \leq n \).
“Square” systems play a huge role in linear algebra:

**Theorem (Systems of \(n\) equations in \(n\) variables)**

A linear system of \(n\) equations (rows in the coefficient matrix) in \(n\) variables (columns in the coefficient matrix) has a unique solution if and only if the rank of the coefficient matrix \(A\) satisfies \(\text{rank}(A) = n\). When that is true the Reduced Row Echelon Form of \(A\) satisfies

\[
\text{rref}(A) = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix}
\]

that is \(\text{rref}(A)\) is the \(n \times n\) identity matrix, usually denoted \(I_n\).
We now define ways that our Matrix and Vector objects can “interact;” we are adding some “verbs” to our Mathematical language!

**Definition (Matrix Sums)**

The sum of two matrices of the same size \( A, B \in \mathbb{R}^{n \times m} \) is determined by the entry-by-entry sums, that is if

\[
C = A + B
\]

then \( C \in \mathbb{R}^{n \times m} \), and \( c_{ij} = a_{ij} + b_{ij} \) for \( i \in [1, \ldots, n], j \in [1, \ldots, m] \).

**Definition (Scalar Multiple of a Matrix)**

If \( A \in \mathbb{R}^{n \times m} \) is a matrix, and \( \rho \in \mathbb{R} \) is a real scalar, then the scalar-matrix-product

\[
C = \rho A
\]

gives \( C \in \mathbb{R}^{n \times m} \), and \( c_{ij} = \rho a_{ij} \).
Definition (Dot product of vectors)

Consider two vectors \( \vec{v} \) and \( \vec{w} \), both with \( n \) components (that is \( v_1, v_2, \ldots, v_n \) and \( w_1, w_2, \ldots, w_n \)). The **dot product** is defined as the sum of the element-wise products:

\[
\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \ldots + v_n w_n = \sum_{k=1}^{n} v_k w_k
\]

**Note:** The way we have defined the dot product it is not row/column sensitive. However, if you stick with the standard notation that “vectors” are column-vectors, it common to see the equivalent notation:

\[
\vec{v}^T \vec{w} \equiv \vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \ldots + v_n w_n = \sum_{k=1}^{n} v_k w_k,
\]

which tends to be referred to as the **inner product.**
Definition (Matrix-Vector product)

If $A \in \mathbb{R}^{n \times m}$ matrix with row-vectors $\vec{r}_1^T, \ldots, \vec{r}_n^T \in \mathbb{R}^m$, and $\vec{x} \in \mathbb{R}^m$ is a (column) vector, then

$$A\vec{x} = \begin{bmatrix} \vec{r}_1^T & \cdots & \vec{r}_n^T \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{r}_1^T \cdot \vec{x} \\ \vdots \\ \vec{r}_n^T \cdot \vec{x} \end{bmatrix} \equiv \begin{bmatrix} \vec{r}_1^T \vec{x} \\ \vdots \\ \vec{r}_n^T \vec{x} \end{bmatrix}$$

The $i^{th}$ component of the resulting vector $\vec{y} = A\vec{x}$ is given by the dot (inner) product of the $i^{th}$ row of $A$ and the vector $\vec{x}$. Note that if $m \neq n$ then $\vec{y} \in \mathbb{R}^n$ is not the same size as $\vec{x} \in \mathbb{R}^m$.

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Size and Shape Do Matter in Matrix-Vector Multiplication

For the matrix-vector product to make sense, the matrix $A \in \mathbb{R}^{n \times m}$ and the vector $\vec{x} \in \mathbb{R}^{m} \equiv \mathbb{R}^{m \times 1}$ must have compatible sizes:

\[
A \begin{bmatrix} n \times m \end{bmatrix} \begin{bmatrix} m \times 1 \end{bmatrix} = \begin{bmatrix} n \times 1 \end{bmatrix} = \vec{y}
\]

Looking Ahead (Matrix Multiplication): thinking about size, it’s probably OK to multiply $A \in \mathbb{R}^{n \times m}$ and $B \in \mathbb{R}^{m \times p}$; a solid “guess” for the size of the result? —

\[
A \begin{bmatrix} n \times m \end{bmatrix} \begin{bmatrix} m \times p \end{bmatrix} = \begin{bmatrix} n \times p \end{bmatrix} = C,
\]

however the product $BA$ does not make sense (unless $n = p$).
Thinking about $A\vec{x}$ in a different way

So far, we have thought of the components of $A\vec{x}$ as the result of dot-products of the rows of $A$ and the vector $\vec{x}$; to inspire a different view:

Consider $A \in \mathbb{R}^{2 \times 3}$ and $\vec{x} \in \mathbb{R}^{3}$, then

$$A\vec{x} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix}$$

We realize that

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$$

Which means that we can think of $\vec{y} = A\vec{x}$ as a sum of vectors (where the vectors are the columns of $A$, scaled by the components of $\vec{x}$)
Thinking about $A\vec{x}$ as the Linear Combination of the Columns

**Theorem (The Product $A\vec{x}$ in Terms of the Columns of $A$)**

If the column vectors of an $n \times m$ matrix $A$ are $\vec{v}_1, \ldots, \vec{v}_m$ and $\vec{x} \in \mathbb{R}^m$ with components $x_1, \ldots, x_m$, then

$$A\vec{x} = \begin{bmatrix} | & | & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & | & | \\ \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1 \vec{v}_1 + \cdots + x_m \vec{v}_m.$$ 

**Definition (Linear Combinations)**

A vector $\vec{b}$ in $\mathbb{R}^n$ is called a **linear combination** of the vectors $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$ if there exists scalars $x_1, \ldots, x_m$ such that

$$\vec{b} = x_1 \vec{v}_1 + \cdots + x_m \vec{v}_m.$$
Let’s just pile ’em on... and let’s do some clean-up and examples in the next set of slides...

Theorem (Algebraic Rules for $A\vec{x}$)

If $A \in \mathbb{R}^{n \times m}$, $\vec{x} \in \mathbb{R}^{m}$, $\vec{y} \in \mathbb{R}^{m}$, and $k \in \mathbb{R}$, then

- $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$
- $A(k\vec{x}) = k(A\vec{x})$

Theorem (Matrix Form of Linear System)

We can write the linear system with Augmented Matrix $[A | \vec{b}]$ in matrix-vector form as

$$A\vec{x} = \vec{b}.$$
Suggested Problems 1.3

Available on “Learning Glass” videos:

1.3.1 Given rref, how many solutions does each system have?
1.3.2 Find the rank of a matrix.
1.3.3 Find the rank of a matrix.
1.3.7 How many solutions? (Geometrical argument).
1.3.13 Compute matrix-vector product.
1.3.22 Given a system + properties of the solution; what is the form of rref(A)?
1.3.23 Given a system + properties of the solution; what is the form of rref(A)?
1.3.37 Find all solutions of $A\vec{x} = \vec{b}$.
1.3.46 Find rank$(A)$.
1.3.55 Is a given vector a linear combination of two other vectors?
<table>
<thead>
<tr>
<th>Lecture</th>
<th>Book, [GS5–]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>§2.2</td>
</tr>
<tr>
<td>1.2</td>
<td>§1.1, §1.3, §2.1, §2.3</td>
</tr>
<tr>
<td>1.3</td>
<td>§1.1, §1.2, §1.3, §2.1, §2.3</td>
</tr>
</tbody>
</table>
### Metacognitive Exercise — Thinking About Thinking & Learning

<table>
<thead>
<tr>
<th>I know / learned</th>
<th>Almost there</th>
<th>Huh?!?</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Right After Lecture</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>After Thinking / Office Hours / SI-session</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>After Reviewing for Midterm/Final</strong></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

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Matrix Algebra
The reduced-row-echelon-forms (RREF) of the augmented matrices of three systems are given. How many solutions does each system have?

(a) \[
\begin{bmatrix}
1 & 0 & 2 & 0 \\
0 & 1 & 3 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\]
(b) \[
\begin{bmatrix}
1 & 0 & 5 \\
0 & 1 & 6 \\
0 & 0 & 1
\end{bmatrix},
\]
(c) \[
\begin{bmatrix}
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 3
\end{bmatrix}.
\]
(1.3.2) Find the rank of

\[ A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \]

(1.3.3) Find the rank of

\[ A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \]
(1.3.7) Consider the vectors $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3 \in \mathbb{R}^2$:

How many solutions $x$, $y$ does the system

$$x\vec{v}_1 + y\vec{v}_2 = \vec{v}_3$$

have? Argue geometrically.
(1.3.13) Compute the matrix-vector product $A\vec{x}$, where

\[ A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}. \]

(1.3.22) Consider a linear system of 3 equations with 3 unknowns, $A\vec{x} = \vec{b}$. GIVEN: This system has a unique solution. What does the reduced-row-echelon-form of the coefficient matrix, $\text{rref}(A)$ of this system look like?

(1.3.23) Consider a linear system of 4 equations with 3 unknowns, $A\vec{x} = \vec{b}$. GIVEN: This system has a unique solution. What does the reduced-row-echelon-form of the coefficient matrix, $\text{rref}(A)$ of this system look like?
(1.3.37) Find all vectors \( \vec{x} \) such that \( A\vec{x} = \vec{b} \), where

\[
A = \begin{bmatrix}
1 & 2 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\
1 \\
0 \end{bmatrix}.
\]

(1.3.46) Find the rank of the matrix

\[
A = \begin{bmatrix}
a & b & c \\
0 & d & e \\
0 & 0 & f
\end{bmatrix},
\]

where \( a, d, f \neq 0 \); and \( b, c, e \in \mathbb{R}^n \) are arbitrary numbers.
(1.3.55) Is the vector
\[
\begin{bmatrix}
7 \\
8 \\
9
\end{bmatrix}
\]
a linear combination of the vectors
\[
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}, \quad \text{and} \quad \begin{bmatrix}
4 \\
5 \\
6
\end{bmatrix}.
\]