Math 254: Introduction to Linear Algebra

Notes #1.3 —
Solutions of Linear systems; Matrix Algebra

Peter Blomgren
⟨blomgren@sdsu.edu⟩

Department of Mathematics and Statistics
Dynamical Systems Group
Computational Sciences Research Center
San Diego State University
San Diego, CA 92182-7720

http://terminus.sdsu.edu/

Spring 2022
(Revised: January 18, 2022)
Outline

1. Student Learning Objectives
   - SLOs: Solutions of Linear systems; Matrix Algebra

2. The Number of Solutions to a System of Linear Equations
   - Collecting the Results... and Adding More Language / Notation
   - Mathematical Language: Logic
   - Using Logic to Derive More Results Re: Variables and Rank

3. Definitions and Rules of Matrix Algebra
   - Fundamentals of Matrix and Vector Algebra

4. Suggested Problems
   - Suggested Problems 1.3
   - Lecture–Book Roadmap

5. Supplemental Material
   - Metacognitive Reflection
   - Problem Statements 1.3

Peter Blomgren (blomgren@sdsu.edu)
After this lecture you should:

- Know what the **Rank** of a Matrix is; and its connection to total/leading/free variables and the number of solutions of a linear system.

- Know the Fundamentals of:
  - Matrix-Vector algebra
  - Vector-Vector Dot Product / Inner Product
  - Matrix-Vector Product: Linear combinations
How Many Solutions Are There?!?

That’s a good question; and it ties in with last lecture...

Let’s ponder the three (augmented, eliminated) systems:

\[
\begin{align*}
a. & \quad \begin{bmatrix}
1 & 2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix} \\
b. & \quad \begin{bmatrix}
1 & 2 & 0 & 1 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0
\end{bmatrix} \\
c. & \quad \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 3
\end{bmatrix}
\end{align*}
\]

Where the leading coefficients have been circled in red. Notice that in a, we did not circle the 1 in the third row, since it belongs to the right-hand-side (and NOT the coefficient matrix).
System a. — No Solutions

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>0</th>
<th>0</th>
<th>( x_1 = 0 - 2x_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>( x_3 = 0 )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( 0 = 1 )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( 0 = 0 )</td>
</tr>
</tbody>
</table>

Here, the third row shows that there are no solutions to this system. We say that the system is inconsistent.
System b. — Infinitely Many Solutions

\[
\begin{bmatrix}
1 & 2 & 0 & 1 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

We are left with one un-determined (free) variable; and introduce a parameter for \(x_2\) (let’s pick the Greek letter \(\eta\) for fun), and write the infinitely many solutions as:

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
1 - 2\eta \\
\eta \\
2
\end{bmatrix} = \begin{bmatrix}
1 \\
0 \\
2
\end{bmatrix} + \eta \begin{bmatrix}
-2 \\
1 \\
0
\end{bmatrix}, \quad \text{where} \quad \eta \in \mathbb{R} \Leftrightarrow \eta \in [\cdots, +\infty]\]
System c. — One (Unique) Solutions

\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 3 \\
\end{bmatrix}
\]

interpretation

\[
\begin{align*}
\mathbf{x}_1 &= 1 \\
\mathbf{x}_2 &= 2 \\
\mathbf{x}_3 &= 3 \\
\end{align*}
\]

Here, there are no un-determined (free) variables; so there’s only one solution.
Consistent vs. Inconsistent Linear Systems

Theorem (Number of Solutions of a Linear System)

A system of equations is said to be **consistent** if there is at least one solution; it is **inconsistent** if there are no solutions.

A linear system is inconsistent if and only if the reduced row-echelon form of its augmented matrix contains the row

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 1
\end{bmatrix},
\]

representing the equation “0 = 1.”

*If a linear system is consistent, then it has either*

- infinitely many solutions, if there is at least one free variable,  
  or
- exactly one solution, if all the variables are leading.*
Definitions

**The Rank of a Matrix**

**Definition (The RANK of a Matrix)**

The **rank** of a matrix is the number of leading 1s in \( \text{rref}(A) \) — the Reduced Row Echelon Form of \( A \) — and is denoted \( \text{rank}(A) \).

**Definition (Full RANK)**

If \( A \in \mathbb{R}^{n \times n} \) (a square matrix of size \( n \)), and \( \text{rank}(A) = n \), then the matrix is said to have **full rank**.

**Heads-up!** In terms of linear systems; the important rank is that of the *coefficient matrix*...
Properties of the $\text{rank}(A)$

Consider a matrix $A \in \mathbb{R}^{n \times m}$, corresponding to a linear system of $n$ equations with $m$ unknowns:

Property #1a, and #1b

The inequalities

$$\text{rank}(A) \leq n, \quad \text{and} \quad \text{rank}(A) \leq m$$

hold.

“Proof:” If we transform $A$ into $\text{rref}(A)$, there is at most one leading 1 in each of the $n$ rows (showing #1a); and there is at most one leading 1 in each of the $m$ columns (showing #1b).
Properties of the \( \text{rank}(A) \)

Property \#2
If the system is inconsistent, then

\[
\text{rank}(A) < n.
\]

“Proof:” For an inconsistent matrix \( A \), \( \text{rref}(A) \) will contain (at least) a row of the form \([ 0 \ 0 \ 0 \ 0 \mid 1 \] — which does not have a leading one — so the rank can be at most \((n - 1)\).
Properties of the \( \text{rank}(A) \)

**Property #3**
If the system has exactly one solution, then

\[
\text{rank}(A) = m.
\]

**Proof:** A leading 1 for each variable leaves no free (un-determined) variables.

**Property #4**
If the system has infinitely many solutions, then

\[
\text{rank}(A) < m.
\]

**Proof:** In this case, there’s at least one free (un-determined) variable, which does not have a corresponding leading 1.
Properties of the $\text{rank}(A)$

It is true that (for $A \in \mathbb{R}^{n \times m}$)

$$\text{#Free\_Variables} = \text{#Total\_Variables} - \text{#Leading\_Variables} = m - \text{rank}(A).$$
More Mathematical Language: The Contrapositive

Definition (The Contrapositive of a Statement)

The contrapositive of a logic statement “if \( p \) then \( q \)”, in math notation: \( p \rightarrow q \); is: “if not-\( q \) then not-\( p \)”, notation: \( (\sim q) \rightarrow (\sim p) \).

The contrapositive of

\[
\begin{align*}
\text{if } & \text{you are in this room} \quad \text{then } \text{you are in this building} \\
& p \quad q
\end{align*}
\]

is

\[
\begin{align*}
\text{if } & \text{you are not in this building} \quad \text{then } \text{you are not in this room} \\
& (\sim q) \quad (\sim p)
\end{align*}
\]

A statement and its contrapositive are logically equivalent; that is if the statement is true, then the contrapositive is true.
Using the Contrapositive

We have some true statements (for $A \in \mathbb{R}^{n \times m}$):

1. if the system is inconsistent, then $\text{rank}(A) < n$.
2. if the system has exactly one solution, then $\text{rank}(A) = m$.
3. if the system has infinitely many solutions, then $\text{rank}(A) < m$.

Using the contrapositive, we immediately can say that

1. if $\text{rank}(A) = n$, then the system is consistent.
2. if $\text{rank}(A) < m$, then the system has either no solutions, or infinitely many solutions.
3. if $\text{rank}(A) = m$, then the system has no solutions, or exactly one solution.
Additional Discussion 1

In all cases below, \( A \in \mathbb{R}^{n \times m} \), \( \text{rank}(A) \leq \min(n, m) \).

For an **inconsistent** system, there must be (as least) one row with zeros one the coefficient-side, and a non-zero on the right-hand-side:

\[
\begin{bmatrix}
\times & \cdots & \times & \times \\
\vdots & & \vdots & \\
0 & \cdots & 0 & 1
\end{bmatrix}
\]

Therefore, \( \text{rank}(A) < n \).
When a system has **exactly one solution**, then \( \text{rref}(A) \) must have a *leading one* in each column (no free variables can remain). The number of columns \( (m) \) equals the number of variables; so we must have \( \text{rank}(A) = m \). Note that therefore \( n \geq m \) — there can only be a single *leading one* in each row. We get two cases:

- \( (n = m) \) \( \Rightarrow \) \( \text{rref}(A) = I_n \)
- \( (n > m) \) \( \Rightarrow \) Rows \( (m+1) \) to \( (n) \) must be all zeros, with zero right-hand-side.

When a system has **infinitely many solutions**, there is at least one *free variable*. Therefore \( \text{rref}(A) \) must have at least one column *without* a leading one, which means that \( \text{rank}(A) \leq (m - 1) \). \( \Rightarrow \) \( \text{rank}(A) < m \).
Additional Discussion III

Thinking about the contrapositive statements...

- When \( \text{rank}(A) = n \), there are leading ones in each row of the reduced system. Therefore, there cannot be any row of the form

\[
\begin{bmatrix}
0 & \cdots & 0 & 1
\end{bmatrix}
\]

which would indicate inconsistency. Hence, the system must be consistent. Again, we have two cases:

- \((m = n) \Rightarrow \text{rref}(A) = I_n \Rightarrow \) the solution is unique.
- \((m > n) \Rightarrow \) there are \((m - n)\) free variables \(\Rightarrow\) there are infinitely many solutions.
Additional Discussion IV

When $\text{rank}(A) < m$, there is at least one column without a leading one $\Rightarrow$ there is at least one free variable. Note that this does not rule out rows of the form

$$\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}.$$ 

if such a row exists, the system is inconsistent and has no solutions, otherwise the system is consistent with (at least) one free variable, and has infinitely many solutions.

When $\text{rank}(A) = m$, there is a leading one in each column $\Rightarrow$ there are no free variables. If there is a row of the form

$$\begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}.$$ 

the system is inconsistent and has no solutions, otherwise the system is consistent with a unique solution.
The number of equations vs. the number of unknowns

**Theorem (#Equations vs. #Ununknowns)**

- **statement:** If a linear system has exactly one solution, then there must be at least as many equations as there are variables; \((m \leq n)\) using previous notation. [The coefficient matrix is either square, or “tall and skinny.”]

- **contrapositive:** If a linear system has fewer equations than unknowns \((n < m)\), then it either has no solutions or infinitely many solutions. [The coefficient matrix is “short and wide.”]

**Proof (of statement).**

A system with exactly one solution has \(m = \text{rank}(A) [\text{PROPERTY } \#3]\); further \(\text{rank}(A) \leq n [\text{PROPERTY } \#1A]\), therefore

\[ m = \text{rank}(A) \leq n \]

which shows \((m \leq n)\).
Square Matrices, and Their Reduced-Row-Echelon-Form

“Square” systems play a huge role in linear algebra:

**Theorem (Systems of $n$ Equations in $n$ Variables)**

A linear system of $n$ equations (rows in the coefficient matrix) in $n$ variables (columns in the coefficient matrix) has a unique solution if and only if the rank of the coefficient matrix $A$ satisfies $\text{rank}(A) = n$. When that is true the Reduced Row Echelon Form of $A$ satisfies

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

that is $\text{rref}(A)$ is the $(n \times n)$ identity matrix, usually denoted $I_n$. 
Fundamentals of Matrix and Vector Algebra

We now define ways that our Matrix and Vector objects can “interact”; we are adding some “verbs” to our Mathematical language!

Definition (Matrix Sums)

The sum of two matrices of the same size $A, B \in \mathbb{R}^{n \times m}$ is determined by the entry-by-entry sums, that is if

$$C = A + B$$

then $C \in \mathbb{R}^{n \times m}$, and $c_{ij} = a_{ij} + b_{ij}$ for $i \in [1, \ldots, n], j \in [1, \ldots, m]$.

Definition (Scalar Multiple of a Matrix)

If $A \in \mathbb{R}^{n \times m}$ is a matrix, and $\rho \in \mathbb{R}$ is a real scalar, then the scalar-matrix-product

$$C = \rho A$$

gives $C \in \mathbb{R}^{n \times m}$, and $c_{ij} = \rho a_{ij}$.
Fundamentals of Matrix and Vector Algebra

Definition (Dot Product of Vectors)

Consider two vectors \( \vec{v} \), and \( \vec{w} \), both with \( n \) components (that is \( v_1, v_2, \ldots, v_n \) and \( w_1, w_2, \ldots, w_n \)). The **dot product** is defined as the sum of the element-wise products:

\[
\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n = \sum_{k=1}^{n} v_k w_k
\]

**Note:** The way we have defined the dot product it is not row/column sensitive. However if you stick with the standard notation that “vectors” are column-vectors, it is common to see the equivalent notation:

\[
\vec{v}^T \vec{w} \equiv \vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + \cdots + v_n w_n = \sum_{k=1}^{n} v_k w_k.
\]

A common alternative name for the dot product, is the **inner product**.

Peter Blomgren (blomgren@sdsu.edu)
Definition (Matrix-Vector Product)

If $A \in \mathbb{R}^{n \times m}$ matrix with row-vectors $\vec{r}_1^T, \ldots, \vec{r}_n^T \in \mathbb{R}^m$, and $\vec{x} \in \mathbb{R}^m$ is a (column) vector, then

$$A\vec{x} = \begin{bmatrix} \vec{r}_1^T & \vdots & \vec{r}_n^T \end{bmatrix} \vec{x} = \begin{bmatrix} \vec{r}_1^T \cdot \vec{x} \\ \vdots \\ \vec{r}_n^T \cdot \vec{x} \end{bmatrix} \equiv \begin{bmatrix} \vec{r}_1^T \vec{x} \\ \vdots \\ \vec{r}_n^T \vec{x} \end{bmatrix}$$

Using Inner Product Notation

Using Dot Product Notation

The $i^{th}$ component of the resulting vector $\vec{y} = A\vec{x}$ is given by the dot (inner) product of the $i^{th}$ row of $A$ and the vector $\vec{x}$. Note that if $m \neq n$ then $\vec{y} \in \mathbb{R}^n$ is not the same size as $\vec{x} \in \mathbb{R}^m$. 

Peter Blomgren (blomgren@sdsu.edu)
Size and Shape Do Matter in Matrix-Vector Multiplication

For the matrix-vector product to make sense, the matrix $A \in \mathbb{R}^{n \times m}$ and the vector $\vec{x} \in \mathbb{R}^{m} \equiv \mathbb{R}^{m \times 1}$ must have compatible sizes:

$$A \begin{bmatrix} n \times m \\ m \times 1 \end{bmatrix} \vec{x} \begin{bmatrix} m \times 1 \end{bmatrix} = \vec{y} \begin{bmatrix} n \times 1 \end{bmatrix}$$

Looking Ahead (Matrix Multiplication): thinking about size, it’s probably OK to multiply $A \in \mathbb{R}^{n \times m}$ and $B \in \mathbb{R}^{m \times p}$; a solid “guess” for the size of the result? —

$$A \begin{bmatrix} n \times m \\ m \times p \end{bmatrix} B \begin{bmatrix} m \times p \end{bmatrix} = C \begin{bmatrix} n \times p \end{bmatrix}$$

however the product $BA$ does not make sense (unless $n = p$).

We will formally define Matrix-Matrix products in [NOTES#3.3].
Thinking About $A\vec{x}$ in a Different Way

So far, we have thought of the components of $A\vec{x}$ as the result of dot-products of the rows of $A$ and the vector $\vec{x}$; to inspire a different view:

Consider $A \in \mathbb{R}^{2 \times 3}$ and $\vec{x} \in \mathbb{R}^3$, then

$$A\vec{x} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix}$$

We realize that

$$\begin{bmatrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} + x_3 \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix}$$

Which means that we can think of $\vec{y} = A\vec{x}$ as a sum of vectors (where the vectors are the columns of $A$, scaled by the components of $\vec{x}$).
Thinking about $A\vec{x}$ as the Linear Combination of the Columns

Theorem (The Product $A\vec{x}$ in Terms of the Columns of $A$)

If the column vectors of an $n \times m$ matrix $A$ are $\vec{v}_1, \ldots, \vec{v}_m$ and $\vec{x} \in \mathbb{R}^m$ with components $x_1, \ldots, x_m$, then

$$A\vec{x} = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_m \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1 \vec{v}_1 + \cdots + x_m \vec{v}_m.$$

Definition (Linear Combinations)

A vector $\vec{b}$ in $\mathbb{R}^n$ is called a **linear combination** of the vectors $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$ if there exists scalars $x_1, \ldots, x_m$ such that

$$\vec{b} = x_1 \vec{v}_1 + \cdots + x_m \vec{v}_m.$$
Challenge Question

Think, again, about the linear systems:

\[
\begin{bmatrix}
1 & 2 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 0 & 1 \\
0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 1
\end{bmatrix}
\]

Let \( A_a \in \mathbb{R}^{4 \times 3}, A_b \in \mathbb{R}^{3 \times 3}, A_c \in \mathbb{R}^{3 \times 3} \) be the coefficient matrices; and \( \vec{b}_a \in \mathbb{R}^4, \vec{b}_b, \vec{b}_c \in \mathbb{R}^3 \) be the right-hand-sides. We are seeking solutions \( \vec{x}_a, \vec{x}_b, \vec{x}_c \in \mathbb{R}^3 \), so that

\[
A_a \vec{x}_a = \vec{b}_a, \quad A_b \vec{x}_b = \vec{b}_b, \quad A_c \vec{x}_c = \vec{b}_c.
\]

If we think of the matrix-vector products as linear combinations of the columns; how can we characterize the 3 possible scenarios (no, \( \infty \), 1) solutions?

Does the rank have anything to do with it?

This will be answered very soon, but do think about it...
Two More Theorems...

Theorem (Algebraic Rules for $A\vec{x}$)

If $A \in \mathbb{R}^{n \times m}$, $\vec{x} \in \mathbb{R}^m$, $\vec{y} \in \mathbb{R}^m$, and $k \in \mathbb{R}$, then

- $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$
- $A(k\vec{x}) = k(A\vec{x})$

Theorem (Matrix Form of Linear System)

We can write the linear system with Augmented Matrix $[A \mid \vec{b}]$ in matrix-vector form as

$$A\vec{x} = \vec{b}.$$
Suggested Problems 1.3

Available on “Learning Glass” videos:

1.3.1 Given \( \text{rref} \), how many solutions does each system have?
1.3.2 Find the rank of a matrix.
1.3.3 Find the rank of a matrix.
1.3.7 How many solutions? (Geometrical argument).
1.3.13 Compute matrix-vector product.
1.3.22 Given a system + properties of the solution; what is the form of \( \text{rref}(A) \)?
1.3.23 Given a system + properties of the solution; what is the form of \( \text{rref}(A) \)?
1.3.37 Find all solutions of \( A\vec{x} = \vec{b} \).
1.3.46 Find \( \text{rank}(A) \).
1.3.55 Is a given vector a linear combination of two other vectors?
### Lecture – Book Roadmap

<table>
<thead>
<tr>
<th>Lecture</th>
<th>Book, [GS5–]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>§2.2</td>
</tr>
<tr>
<td>1.2</td>
<td>§1.1, §1.3, §2.1, §2.3</td>
</tr>
<tr>
<td>1.3</td>
<td>§1.1, §1.2, §1.3, §2.1, §2.3</td>
</tr>
</tbody>
</table>
Metacognitive Exercise — Thinking About Thinking & Learning

- I know / learned
- Almost there
- Huh?!?

Right After Lecture

After Thinking / Office Hours / SI-session

After Reviewing for Quiz/Midterm/Final
The reduced-row-echelon-forms (RREF) of the augmented matrices of three systems are given. How many solutions does each system have?

\[
\begin{align*}
(a) & \quad \begin{bmatrix} 1 & 0 & 2 & | & 0 \\ 0 & 1 & 3 & | & 0 \\ 0 & 0 & 0 & | & 1 \end{bmatrix}, &
(b) & \quad \begin{bmatrix} 1 & 0 & | & 5 \\ 0 & 1 & | & 6 \end{bmatrix}, &
(c) & \quad \begin{bmatrix} 0 & 1 & 0 & | & 2 \\ 0 & 0 & 1 & | & 3 \end{bmatrix}.
\end{align*}
\]
(1.3.2), (1.3.3)

(1.3.2) Find the rank of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

(1.3.3) Find the rank of

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$
Consider the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^2$:

How many solutions $x, y$ does the system

$$x\vec{v}_1 + y\vec{v}_2 = \vec{v}_3$$

have? Argue geometrically.
(1.3.13), (1.3.22), (1.3.23)

(1.3.13) Compute the matrix-vector product $A\vec{x}$, where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} 7 \\ 11 \end{bmatrix}. $$

(1.3.22) Consider a linear system of 3 equations with 3 unknowns, $A\vec{x} = \vec{b}$. GIVEN: This system has a unique solution. What does the reduced-row-echelon-form of the coefficient matrix, $\text{rref}(A)$ of this system look like?

(1.3.23) Consider a linear system of 4 equations with 3 unknowns, $A\vec{x} = \vec{b}$. GIVEN: This system has a unique solution. What does the reduced-row-echelon-form of the coefficient matrix, $\text{rref}(A)$ of this system look like?
(1.3.37), (1.3.46)

(1.3.37) Find all vectors \( \vec{x} \) such that \( A \vec{x} = \vec{b} \), where

\[
A = \begin{bmatrix}
1 & 2 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}, \quad \vec{b} = \begin{bmatrix}
2 \\
1 \\
0
\end{bmatrix}.
\]

(1.3.46) Find the rank of the matrix

\[
A = \begin{bmatrix}
a & b & c \\
0 & d & e \\
0 & 0 & f
\end{bmatrix},
\]

where \( a, d, f \neq 0 \); and \( b, c, e \in \mathbb{R}^n \) are arbitrary numbers.
Is the vector \[
\begin{bmatrix}
7 \\
8 \\
9
\end{bmatrix}
\] a linear combination of the vectors \[
\begin{bmatrix}
1 \\
2 \\
3
\end{bmatrix}, \quad \text{and} \quad \begin{bmatrix}
4 \\
5 \\
6
\end{bmatrix}.
\]