Student Learning Objectives

SLOs: Inverse of a Linear Transform

After this lecture you should:

- Know the connection between invertibility, the form of \( \text{rref}(A) \), and the rank of \( A \):
  \[ A \in \mathbb{R}^{n \times n} \text{ is invertible } \iff \text{rref}(A) = I_n \iff \text{rank}(A) = n. \]
- Know the definition of, and be able to compute the kernel of a matrix: \( \ker(A) = \{ \vec{x} : A\vec{x} = \vec{0} \} \)
- Given an invertible matrix \( A \), perform row-operations to find the inverse (see also [Notes#2.1]):
  \[
  \begin{bmatrix}
  A & I_n \\
  \end{bmatrix} \sim
  \begin{bmatrix}
  I_n & A^{-1} \\
  \end{bmatrix}
  \]
- Be able to compute the determinant, \( \det(A) \) and know its Geometrical Interpretation for \( A \in \mathbb{R}^{2 \times 2} \).
Inverse of a Linear Transformation
Suggested Problems
Invertible Functions
Invertible Linear Transformations and Matrices

[FOCUS :: MATH] “Speak” Like a Mathematician

We can also say that a function is invertible if and only if it is both “onto” (surjective) and “1-to-1” (injective).

Definition (One-to-One Function [adopted from Wikipedia])
In mathematics, an injective function or injection or one-to-one function is a function that preserves distinctness: it never maps distinct elements of its domain to the same element of its range. In other words, every element of the function’s range is the image of at most one element of its domain. The term one-to-one function must not be confused with one-to-one correspondence (a.k.a. bijective function), which uniquely maps all elements in both domain and range to each other.

Definition (Onto Function [adopted from Wikipedia])
In mathematics, a function $f$ from a set $X$ to a set $Y$ is surjective (or onto), or a surjection, if for every element $y$ in the range $Y$ of $f$ there is at least one element $x$ in the domain $X$ of $f$ such that $f(x) = y$. It is not required that $x$ be unique; the function $f$ may map one or more elements of $X$ to the same element of $Y$.

Invertible Functions

Example ($f$ and its inverse $f^{-1}$)

Let

$$f(x) = \frac{x^5 - 1}{3}, \quad g(y) = \sqrt[3]{3y + 1}$$

with $x \in [0, \infty)$, and $y \in \left[-\frac{1}{3}, \infty\right)$. Then

$$f(g(y)) = f\left(\sqrt[3]{3y + 1}\right) = \frac{\left(\sqrt[3]{3y + 1}\right)^5 - 1}{3} = y,$$

and

$$g(f(x)) = g\left(\frac{x^5 - 1}{3}\right) = \sqrt[3]{3\frac{x^5 - 1}{3} + 1} = x.$$
Inverse of a Linear Transformation

Characteristics of Invertible Matrices

**Definition (Invertible Matrices, $A \mapsto A^{-1}$)**

A square matrix $A$ is said to be **invertible** if the linear transformation $\vec{y} = T(\vec{x}) = A\vec{x}$ is invertible. In this case the matrix of $T^{-1}$ is denoted $A^{-1}$. If the linear transformation $\vec{y} = T(\vec{x}) = A\vec{x}$ is invertible, then its inverse is $\vec{x} = T^{-1}(\vec{y}) = A^{-1}\vec{y}$.

**Theorem (Invertibility, Rank, and RREF)**

An $n \times n$ matrix $A$ is invertible if and only if

$$\text{rref}(A) = I_n$$

or, equivalently, if and only if

$$\text{rank}(A) = n.$$  

**Important!**

Equivalent Statements: Invertible Matrices

For an $n \times n$ matrix $A$, the following statements are equivalent; that is for a given $A$, they are either all true or all false:

i. $A$ is invertible ($\exists A^{-1}$)

ii. The linear system $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$, $\forall \vec{b} \in \mathbb{R}^n$

iii. $\text{rref}(A) = I_n$

iv. $\text{rank}(A) = n$

We will add to this list throughout the semester: [Notes#3.1], [Notes#3.3], and [Notes#7.1].

Invertibility and Linear Systems

Recasting some of the results from [Notes#1.3] into our new “language:”

**Theorem (Invertibility and Linear Systems)**

Let $A \in \mathbb{R}^{n \times n}$:

a. Consider a vector $\vec{b} \in \mathbb{R}^n$. If $A$ is invertible, then the system $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$. If $A$ is non-invertible, the system $A\vec{x} = \vec{b}$ has either infinitely many solutions, or no solutions. — The collection of all vectors $\{ \vec{x} : A\vec{x} = \vec{0} \}$ is called the **null space** or **kernel** of $A$, denoted $\ker(A)$.

b. Consider the special case when $\vec{b} = \vec{0}$. The system $A\vec{x} = \vec{0}$ has $\vec{x} = \vec{0}$ as a solution. If $A$ is invertible, then this is the only solution; otherwise it has infinitely many solutions. — The collection of all vectors $\{ \vec{x} : A\vec{x} = \vec{0} \}$ is called the **null space** or **kernel** of $A$, denoted $\ker(A)$.

We will discuss the null space / kernel extensively in the next four lectures (after the midterm).

Finding $A^{-1}$...

**Theorem (Finding the Inverse of a Matrix)**

To find the inverse of and $n \times n$ matrix, form the $n \times (2n)$ matrix

$$\begin{bmatrix} A & I_n \end{bmatrix}$$

and compute

$$\text{rref} \left( \begin{bmatrix} A & I_n \end{bmatrix} \right)$$

- If $\text{rref} \left( \begin{bmatrix} A & I_n \end{bmatrix} \right) = \begin{bmatrix} I_n & B \end{bmatrix}$, then $A$ is invertible, and $A^{-1} = B$.
- If $\text{rref} \left( \begin{bmatrix} A & I_n \end{bmatrix} \right)$ is of another form, then $A$ is not invertible.

**Note:** The best way to establish whether a matrix is invertible is to try to compute the inverse using the method above. If successful, you have $A^{-1}$; otherwise you know that $A$ is not invertible.
Example (Computation of the Matrix Inverse)

Start with \[
\begin{bmatrix}
1 & 1 & 1 \\
2 & 3 & 2 \\
3 & 8 & 2
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

Eliminate Column#1:

\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

Eliminate Column#2:

\[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
3 & -1 & 0 \\
-2 & 1 & 0 \\
7 & -5 & 1
\end{bmatrix}
\]

We arrive at \[
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

The Inverse of the Product of Invertible Matrices

Let \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times n} \) be two invertible matrices; i.e. \( A^{-1} \in \mathbb{R}^{n \times n} \) and \( B^{-1} \in \mathbb{R}^{n \times n} \) exists.

Does \((BA)^{-1}\) exist as well?

<table>
<thead>
<tr>
<th>Linear Transform</th>
<th>( \tilde{y} = BA\tilde{x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiply by ( B^{-1} ) from the left</td>
<td>( B^{-1}\tilde{y} = B^{-1}BA\tilde{x} )</td>
</tr>
<tr>
<td>Multiply by ( A^{-1} ) from the left</td>
<td>( A^{-1}B^{-1}\tilde{y} = A^{-1}A\tilde{x} )</td>
</tr>
<tr>
<td>Inverse Linear Transform</td>
<td>( A^{-1}B^{-1}\tilde{y} = \tilde{x} )</td>
</tr>
</tbody>
</table>
The Inverse of the Product of Invertible Matrices

**Theorem (The Inverse of a Product of Matrices)**

If $A$ and $B$ are invertible $n \times n$ matrices, then $BA$ ($AB$) is invertible as well, and

$$(BA)^{-1} = A^{-1}B^{-1}, \quad (AB)^{-1} = B^{-1}A^{-1}$$

La soupe à l’alphabet:

$$A^{-1}B^{-1}BA = A^{-1}I_nA = A^{-1}A = I_n$$
$$BA A^{-1}B^{-1} = BI_nB^{-1} = BB^{-1} = I_n$$
$$B^{-1}A^{-1}AB = B^{-1}I_nB = B^{-1}B = I_n$$
$$AB B^{-1}A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$$

Example (Inverse of a “Chain” Product of Matrices)

Ponder the invertible matrices $A, B, C, D, E \in \mathbb{R}^{n \times n}$; lets find $C$ and $C^{-1}$ in terms of the other matrices...

<table>
<thead>
<tr>
<th>GIVEN</th>
<th>ABCDE = $I_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$A^{-1}E^{-1}$</td>
</tr>
<tr>
<td>$C$</td>
<td>$A^{-1}C^{-1}AB$</td>
</tr>
<tr>
<td></td>
<td>$C^{-1}B^{-1}A^{-1}I_n$</td>
</tr>
<tr>
<td>$C^{-1}$</td>
<td>$C^{-1}$</td>
</tr>
</tbody>
</table>

Inverse and Determinant

It is true that if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \text{then} \quad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

under the condition that $(ad - bc) \neq 0$.

The quantity $(ad - bc)$ is the determinant of $A$, denoted $\det(A)$.

(We will return to this quantity for general $n \times n$ matrices later.)

Formulas for the inverse (using the determinant) generalize poorly to higher dimensions, check out the next slide for the result in $3 \times 3$ case...
Inverse and Determinant in the $3 \times 3$ Case

Let

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & j \end{bmatrix},$$

then

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} e j - f h & c h - b j & b f - c e \\ f g - d j & a j - c g & c d - a f \\ d h - e g & b g - a h & a e - b d \end{bmatrix},$$

where $\det(A) = bfg - afh + cdh - ceg + aej - bdj$.

Maybe not something you want to try to memorize?

Geometrically Interpreting the Determinant in 2D

Theorem (Geometrical Interpretation of $\det(A)$, for $A \in \mathbb{R}^{2 \times 2}$)

If $A = \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix}$ is a $2 \times 2$ matrix with non-zero column vectors $\vec{v}$ and $\vec{w}$, then

$$\det(A) = \det \left( \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix} \right) = \sin \theta \|v\| \|w\|,$$

where $\theta$ is the angle from $\vec{v}$ to $\vec{w}$, with $\theta \in (-\pi, \pi)$. It follows that:

- $|\det(A)| = |\sin \theta| \|v\| \|w\|$ is the area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$.
- $\det(A) = 0$ if $\vec{v}$ and $\vec{w}$ are parallel, i.e. $\theta = 0$, or $\theta = \pi$.
- $\det(A) > 0$ if $\theta \in (0, \pi)$.
- $\det(A) < 0$ if $\theta \in (-\pi, 0)$.

Example ($2 \times 2$ Determinant: Computed 2 Ways)

Let

$$\vec{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Then

$$\|\vec{v}\| = \sqrt{17}, \quad \|\vec{w}\| = \sqrt{13}$$

$$\det \left( \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix} \right) = \sin(0.737815\ldots)\sqrt{17}\sqrt{13} = 10$$

$$= 4 \cdot 3 - 2 \cdot 1 = 12 - 2 = 10$$

$$\det \left( \begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix} \right) = \sin(-0.737815\ldots)\sqrt{17}\sqrt{13} = -10.$$
2.4. Inverse of a Linear Transform

Suggested Problems

Lecture Book, [GS5–]

<table>
<thead>
<tr>
<th>Lecture</th>
<th>Book, [GS5–]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>§2.2</td>
</tr>
<tr>
<td>1.2</td>
<td>§1.2, §1.3, §2.1, §2.3</td>
</tr>
<tr>
<td>1.3</td>
<td>§1.1, §1.3, §2.1, §2.3</td>
</tr>
<tr>
<td>1.4</td>
<td>§1.2, §1.3, §2.1, §2.3</td>
</tr>
<tr>
<td>2.1</td>
<td>§8.1, §8.2*, §2.5*</td>
</tr>
<tr>
<td>2.2</td>
<td>§8.1, §8.2*, §4.2*, §4.4*</td>
</tr>
<tr>
<td>2.3</td>
<td>§2.4</td>
</tr>
<tr>
<td>2.4</td>
<td>§2.5</td>
</tr>
</tbody>
</table>

§2.5* (p.86–88) “Calculating $A^{-1}$ by Gauss-Jordan Elimination”

§4.2* (p.207) “Projection Onto a Line” – (p.210) end of “Example 2”

§4.4* Example 1, Example 3

§8.2* We will talk about “Basis” / “Bases” soon... don’t worry about those concepts... yet.

Peter Blomgren ⟨blomgren@sdsu.edu⟩

2.4. Inverse of a Linear Transform — (25/33)

Supplemental Material

Metacognitive Exercise — Thinking About Thinking & Learning

I know / learned | Almost there | Huh?!

Right After Lecture

After Thinking / Office Hours / SI-session

After Reviewing for Quiz/Midterm/Final

Suggested Problems

Lecture – Book Roadmap

Live Math Fall 2019 — Projections and Reflections

1 of 2

Given

$$\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \quad \vec{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{w}_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \quad L_1 = \{ k_1 \vec{w}_1 : k_1 \in \mathbb{R} \}
$$

$$L_2 = \{ k_2 \vec{w}_2 : k_2 \in \mathbb{R} \}$$

compute $\text{proj}_{L_1}(\vec{x})$, $\text{proj}_{L_2}(\vec{x})$, $\text{ref}_{L_1}(\vec{x})$, and $\text{ref}_{L_2}(\vec{x})$.

$$\text{proj}_{L_1}(\vec{x}) = \begin{bmatrix} \vec{x} \cdot \vec{w}_1 \\ \vec{w}_1 \cdot \vec{w}_1 \end{bmatrix} = \begin{bmatrix} 1 + 2 + 3 + 4 + 5 = 15 \\ 1 + 1 + 1 + 1 + 1 = 5 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 1/5 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 3/5 \end{bmatrix}$$

$$\text{proj}_{L_2}(\vec{x}) = \begin{bmatrix} \vec{x} \cdot \vec{w}_2 \\ \vec{w}_2 \cdot \vec{w}_2 \end{bmatrix} = \begin{bmatrix} 1 - 2 + 3 - 4 + 5 = 3 \\ 1 + 1 + 1 + 1 + 1 = 5 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 3/5 \end{bmatrix}$$

$$\text{ref}_{L_1}(\vec{x}) = 2\text{proj}_{L_1}(\vec{x}) - \vec{x} = \begin{bmatrix} 1/3 \\ 1/3 \end{bmatrix} - \begin{bmatrix} 5/3 \\ 5/3 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \end{bmatrix}$$

$$\text{ref}_{L_2}(\vec{x}) = 2\text{proj}_{L_2}(\vec{x}) - \vec{x} = \begin{bmatrix} 3/5 \\ -3/5 \end{bmatrix} - \begin{bmatrix} 5/3 \\ 5/3 \end{bmatrix} = \begin{bmatrix} -16/15 \\ -9/15 \end{bmatrix}$$

Peter Blomgren ⟨blomgren@sdsu.edu⟩

2.4. Inverse of a Linear Transform — (26/33)

Supplemental Material

Metacognitive Reflection

Live Math

Problem Statements 2.4

Live Math Fall 2019 — Projections and Reflections

2 of 2

$$\text{proj}_{L_2}(\vec{x}) = \begin{bmatrix} 3/5 \\ -3/5 \end{bmatrix}$$

$$\text{ref}_{L_1}(\vec{x}) = 2\text{proj}_{L_1}(\vec{x}) - \vec{x} = \begin{bmatrix} 1/3 \\ -2/3 \end{bmatrix}$$

$$\text{ref}_{L_2}(\vec{x}) = 2\text{proj}_{L_2}(\vec{x}) - \vec{x} = \begin{bmatrix} -16/15 \\ -9/15 \end{bmatrix}$$

Peter Blomgren ⟨blomgren@sdsu.edu⟩

2.4. Inverse of a Linear Transform — (27/33)

Supplemental Material

Metacognitive Reflection

Live Math

Problem Statements 2.4

Live Math Fall 2019 — Projections and Reflections

Peter Blomgren ⟨blomgren@sdsu.edu⟩

2.4. Inverse of a Linear Transform — (28/33)
(2.4.1), (2.4.3)

(2.4.1) Decide whether the matrix is invertible; if it is, find the inverse.
\[ A = \begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix} \]

(2.4.3) Decide whether the matrix is invertible; if it is, find the inverse.
\[ A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \]

(2.4.9), (2.4.16)

(2.4.9) Decide whether the matrix is invertible; if it is, find the inverse.
\[ A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \]

(2.4.16) Decide whether the linear transformation is invertible; if it is, find the inverse transformation.
\[ \begin{align*}
y_1 &= 3x_1 + 5x_2 \\
y_2 &= 5x_1 + 8x_2
\end{align*} \]

(2.4.17), (2.4.29)

(2.4.17) Decide whether the linear transformation is invertible; if it is, find the inverse transformation.
\[ \begin{align*}
y_1 &= x_1 + 2x_2 \\
y_2 &= 4x_1 + 8x_2
\end{align*} \]

(2.4.29) For which values of the constant \( k \) is the following matrix invertible?
\[ A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{bmatrix} \]

(2.4.31)

(2.4.31) For which values of the constants \( a, b, \) and \( c \) is the following matrix invertible?
\[ A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix} \]
(2.4.35)

(2.4.35)

a. Consider the upper triangular matrix

\[
A = \begin{bmatrix}
a & b & c \\
0 & d & e \\
0 & 0 & f
\end{bmatrix}
\]

For which values of \(a, b, c, d, e,\) and \(f\) is \(A\) invertible?

b. More generally, when is an upper triangular matrix (of size \(n \times n\)) invertible?

c. If an upper triangular matrix is invertible, is its inverse also upper triangular matrix?

d. Repeat questions b. and c. for lower triangular matrices.