Outline

1. Student Learning Objectives
   • SLOs: Inverse of a Linear Transform

2. Inverse of a Linear Transformation
   • Invertible Functions
   • Invertible Linear Transformations and Matrices

3. Suggested Problems
   • Suggested Problems 2.4
   • Lecture–Book Roadmap

4. Supplemental Material
   • Metacognitive Reflection
   • Problem Statements 2.4

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After this lecture you should:

- Know the connection between invertibility, the form of \( \text{rref}(A) \), and the rank of \( A \):
  
  \[
  A \in \mathbb{R}^{n \times n} \text{ is invertible } \iff \text{rref}(A) = I_n \iff \text{rank}(A) = n.
  \]

- Know the definition of, and be able to compute the kernel of a matrix: \( \text{ker}(A) = \{ \vec{x} : A\vec{x} = \vec{0} \} \)

- Given an invertible matrix \( A \), perform row-operations to find the inverse (see also [Lecture 2.1]):

\[
\begin{bmatrix}
A & I_n \\
\end{bmatrix} \sim 
\begin{bmatrix}
I_n & A^{-1} \\
\end{bmatrix}
\]

- Be able to compute the determinant, \( \text{det}(A) \) and know its Geometrical Interpretation for \( A \in \mathbb{R}^{2 \times 2} \).
Invertible Functions

A function \( f : X \rightarrow Y \) is called **invertible** if the equation \( f(x) = y \) has a unique solution \( x \in X \) for each \( y \in Y \). In this case, the inverse \( f^{-1} : Y \rightarrow X \) is defined by

\[
f^{-1}(y) = \{ \text{the unique } x \in X \text{ such that } f(x) = y \}.
\]

**Figure:** \( T \) is invertible since there is a unique \( x \in X \) for each \( y \in Y \); \( S \) is not invertible since there is one \( y \) which is not “reachable” from \( X \); and, finally, \( R \) is not invertible since there is one \( y \) for which \( R(x) = y \) has two solutions.

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Inverse of a Linear Transform
We can also say that a function is invertible if and only if it is both “onto” (surjective) and “1-to-1” (injective).

Definition (One-to-One Function [adopted from Wikipedia])
In mathematics, an injective function or injection or one-to-one function is a function that preserves distinctness: it never maps distinct elements of its domain to the same element of its range. In other words, every element of the function’s range is the image of at most one element of its domain. The term one-to-one function must not be confused with one-to-one correspondence (a.k.a. bijective function), which uniquely maps all elements in both domain and range to each other.

Definition (Onto Function [adopted from Wikipedia])
In mathematics, a function \( f \) from a set \( X \) to a set \( Y \) is surjective (or onto), or a surjection, if for every element \( y \) in the range \( Y \) of \( f \) there is at least one element \( x \) in the domain \( X \) of \( f \) such that \( f(x) = y \). It is not required that \( x \) be unique; the function \( f \) may map one or more elements of \( X \) to the same element of \( Y \).
**Inverse of a Linear Transformation**

**Invertible Functions**

The equation \( x = f^{-1}(y) \) means that \( y = f(x) \).

It is true that \( \forall x \in X \text{ and } \forall y \in Y \)

\[
f^{-1}(f(x)) = x, \quad \text{and} \quad f(f^{-1}(y)) = y
\]

**Calculus Review**

**Inverse of the Inverse**

If \( f \) is invertible, then so is \( f^{-1} \), and \((f^{-1})^{-1} = f\).
Let
\[ f(x) = \frac{x^5 - 1}{3}, \quad g(y) = \frac{5\sqrt[5]{3y + 1} - 1}{3} \]
with \( x \in [0, \infty) \), and \( y \in \left[-\frac{1}{3}, \infty\right) \).

Then
\[ f(g(y)) = f\left(\frac{5\sqrt[5]{3y + 1}}{3}\right) = \frac{(\sqrt[5]{3y + 1})^5 - 1}{3} = y, \]
and
\[ g(f(x)) = g\left(\frac{x^5 - 1}{3}\right) = \sqrt[5]{3\left(\frac{x^5 - 1}{3}\right) + 1} = x. \]
Next, consider a linear transformation \( T : \mathbb{R}^n \rightarrow \mathbb{R}^n \) given by
\[
\vec{y} = T(\vec{x}) = A\vec{x},
\]
here \( A \in \mathbb{R}^{n \times n} \).

The linear transformation is invertible if and only if the linear system
\[
A\vec{x} = \vec{y}
\]
has a unique solution \( \vec{x} \in \mathbb{R}^n \) \( \forall \vec{y} \in \mathbb{R}^n \).

This is true if and only if \( \text{rank}(A) = n \), or equivalently if and only if
\[
\text{rref}(A) = I_n.
\]
Definition (Invertible Matrices, $A \rightarrow A^{-1}$)

A square matrix $A$ is said to be invertible if the linear transformation $\vec{y} = T(\vec{x}) = A\vec{x}$ is invertible. In this case the matrix of $T^{-1}$ is denoted $A^{-1}$. If the linear transformation $\vec{y} = T(\vec{x}) = A\vec{x}$ is invertible, then its inverse is $\vec{x} = T^{-1}(\vec{y}) = A^{-1}\vec{y}$.

Theorem (Invertibility)

An $n \times n$ matrix $A$ is invertible if and only if

$$\text{rref}(A) = I_n$$

or, equivalently, if and only if

$$\text{rank}(A) = n.$$
Recasting some of the results from [Lecture 1.3] into our new “language:”

Theorem (Invertibility and Linear Systems)

Let $A \in \mathbb{R}^{n \times n}$:

a. Consider a vector $\vec{b} \in \mathbb{R}^n$. If $A$ is invertible, then the system $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$. If $A$ is non-invertible, the system $A\vec{x} = \vec{b}$ has either infinitely many solutions, or no solutions.

b. Consider the special case when $\vec{b} = \vec{0}$. The system $A\vec{x} = \vec{0}$ has $\vec{x} = \vec{0}$ as a solution. If $A$ is invertible, then this is the only solution; otherwise it has infinitely many solutions. — The collection of all vectors $\{ \vec{x} : A\vec{x} = \vec{0} \}$ is called the null space or kernel of $A$, denoted $\ker(A)$.

We will discuss the null space/kernel extensively in the next four lectures (after the midterm).
Theorem (Finding the Inverse of a Matrix)

To find the inverse of an $n \times n$ matrix, form the $n \times (2n)$ matrix

$$\begin{bmatrix} A & I_n \end{bmatrix}$$

and compute

$$\text{rref} \left( \begin{bmatrix} A & I_n \end{bmatrix} \right)$$

- If $\text{rref} \left( \begin{bmatrix} A & I_n \end{bmatrix} \right) = \begin{bmatrix} I_n & B \end{bmatrix}$, then $A$ is invertible, and $A^{-1} = B$.
- If $\text{rref} \left( \begin{bmatrix} A & I_n \end{bmatrix} \right)$ is of another form, then $A$ is not invertible.
Example!

Start with \([ A \mid I_3 ]\)

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
2 & 3 & 2 & 0 & 1 & 0 \\
3 & 8 & 2 & 0 & 0 & 1
\end{bmatrix}
\]

Eliminate Column#1:

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & -2 & 1 & 0 \\
0 & 5 & -1 & -3 & 0 & 1
\end{bmatrix}
\]

Eliminate Column#2:

\[
\begin{bmatrix}
1 & 0 & 1 & 3 & -1 & 0 \\
0 & 1 & 0 & -2 & 1 & 0 \\
0 & 0 & -1 & 7 & -5 & 1
\end{bmatrix}
\]
Example!

Normalize Row #3 (Divide by $-1$):

$$
\begin{bmatrix}
1 & 0 & 1 & 3 & -1 & 0 \\
0 & 1 & 0 & -2 & 1 & 0 \\
0 & 0 & 1 & -7 & 5 & -1
\end{bmatrix}
$$

Eliminate Column#3:

$$
\begin{bmatrix}
1 & 0 & 0 & 10 & -6 & 1 \\
0 & 1 & 0 & -2 & 1 & 0 \\
0 & 0 & 1 & -7 & 5 & -1
\end{bmatrix}
$$

We arrive at $[I_3 \mid A^{-1}]$
Inverse of a Linear Transformation

Multiplying by the Inverse

**Theorem**

*For an invertible matrix* $A \in \mathbb{R}^{n \times n}$,

$$A^{-1}A = AA^{-1} = I_n$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 2 \\ 3 & 8 & 2 \end{bmatrix}, \quad A^{-1} = \begin{bmatrix} 10 & -6 & 1 \\ -2 & 1 & 0 \\ -7 & 5 & -1 \end{bmatrix}$$

$$AA^{-1} = \begin{bmatrix} [1, 1, 1] \cdot [10, -2, -7] & [1, 1, 1] \cdot [-6, 1, 5] & [1, 1, 1] \cdot [1, 0, -1] \\ [2, 3, 2] \cdot [10, -2, -7] & [2, 3, 2] \cdot [-6, 1, 5] & [2, 3, 2] \cdot [1, 0, -1] \\ [3, 8, 2] \cdot [10, -2, -7] & [3, 8, 2] \cdot [-6, 1, 5] & [3, 8, 2] \cdot [1, 0, -1] \end{bmatrix} = I_3$$

$$A^{-1}A = \begin{bmatrix} [10, -6, 1] \cdot [1, 2, 3] & [10, -6, 1] \cdot [1, 3, 8] & [10, -6, 1] \cdot [1, 2, 2] \\ [-2, 1, 0] \cdot [1, 2, 3] & [-2, 1, 0] \cdot [1, 3, 8] & [-2, 1, 0] \cdot [1, 2, 2] \\ [-7, 5, -1] \cdot [1, 2, 3] & [-7, 5, -1] \cdot [1, 3, 8] & [-7, 5, -1] \cdot [1, 2, 2] \end{bmatrix} = I_3$$

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The Inverse of the Product of Invertible Matrices

Let $A \in \mathbb{R}^{n \times n}$, and $B \in \mathbb{R}^{n \times n}$ be two invertible matrices; i.e. $A^{-1} \in \mathbb{R}^{n \times n}$ and $B^{-1} \in \mathbb{R}^{n \times n}$ exists.

Does $(BA)^{-1}$ exist as well?

<table>
<thead>
<tr>
<th>LINEAR TRANSFORM</th>
<th>( \vec{y} = B A \vec{x} )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>MULTIPLY BY</strong></td>
<td>( B^{-1} \vec{y} = B^{-1} B A \vec{x} )</td>
</tr>
<tr>
<td><strong>B^{-1} FROM THE</strong></td>
<td>( B^{-1} \vec{y} = A \vec{x} )</td>
</tr>
<tr>
<td><strong>LEFT</strong></td>
<td>( A^{-1} B^{-1} \vec{y} = A^{-1} A \vec{x} )</td>
</tr>
<tr>
<td><strong>MULTIPLY BY</strong></td>
<td>( A^{-1} B^{-1} \vec{y} = \vec{x} )</td>
</tr>
<tr>
<td><strong>A^{-1} FROM THE</strong></td>
<td><strong>LEFT</strong></td>
</tr>
</tbody>
</table>
The Inverse of the Product of Invertible Matrices

**Theorem (The Inverse of a Product of Matrices)**

*If A and B are invertible \( n \times n \) matrices, then \( BA \) (\( AB \)) is invertible as well, and*

\[
(BA)^{-1} = A^{-1}B^{-1}, \quad (AB)^{-1} = B^{-1}A^{-1}
\]

*Proof:

\[
A^{-1}B^{-1} BA = A^{-1}I_nA = A^{-1}A = I_n
\]

\[
BA A^{-1}B^{-1} = B I_n B^{-1} = B B^{-1} = I_n
\]

\[
B^{-1}A^{-1} AB = B^{-1}I_nB = B^{-1}B = I_n
\]

\[
AB B^{-1}A^{-1} = A I_n A^{-1} = AA^{-1} = I_n
\]
Theorem (Invertibility Criterion)

Let $A$ and $B$ be two $n \times n$ matrices such that

$$BA = I_n.$$ 

Then

a. $A$ and $B$ are both invertible
b. $A^{-1} = B$, and $B^{-1} = A$, and
c. $AB = I_n$

We can use this result repeatedly to navigate thru long matrix products...
Example

Ponder the invertible matrices $A, B, C, D, E \in \mathbb{R}^{n \times n}$; lets find $C$ and $C^{-1}$ in terms of the other matrices...

<table>
<thead>
<tr>
<th>GIVEN</th>
<th>$ABCDE = I_n$</th>
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</thead>
<tbody>
<tr>
<td>$C$</td>
<td>$A^{-1}ABCDDE^{-1} = A^{-1}E^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$BCD = A^{-1}E^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$B^{-1}BCDD^{-1} = B^{-1}A^{-1}E^{-1}D^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$C = B^{-1}A^{-1}E^{-1}D^{-1}$</td>
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<tr>
<td>$C^{-1}$</td>
<td>$C^{-1}B^{-1}A^{-1}ABCDE = C^{-1}B^{-1}A^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$DE = C^{-1}B^{-1}A^{-1}$</td>
</tr>
<tr>
<td></td>
<td>$DEAB = C^{-1}B^{-1}A^{-1}AB$</td>
</tr>
<tr>
<td></td>
<td>$DEAB = C^{-1}$</td>
</tr>
</tbody>
</table>
It is true that if

\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ then } A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \]

under the condition that \( ad - bc \neq 0 \). The quantity \( ad - bc \) is the determinant of \( A \), denoted \( \det(A) \). (We will return to this quantity for general \( n \times n \) matrices later.)

Formulas for the inverse (using the determinant) generalize poorly to higher dimensions, check out the next slide for the result in 3D...
Let

\[
A = \begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & j \\
\end{bmatrix},
\]

then

\[
A^{-1} = \frac{1}{\det(A)} \begin{bmatrix}
e j - f h & c h - b j & b f - c e \\
f g - d j & a j - c g & c d - a f \\
d h - e g & b g - a h & a e - b d \\
\end{bmatrix},
\]

where \( \det(A) = bfg - afh + cdh - ceg + aej - bdj \).

Maybe **not** something you want to try to memorize?

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Interpreting the Determinant in 2D

Theorem (Geometrical Interpretation of $\det(A)$, for $A \in \mathbb{R}^{2 \times 2}$)

If $A = \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix}$ is a $2 \times 2$ matrix with non-zero column vectors $\vec{v}$ and $\vec{w}$, then

$$\det(A) = \det \left( \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix} \right) = \sin \theta \|v\| \|w\|,$$

where $\theta$ is the angle from $\vec{v}$ to $\vec{w}$, with $\theta \in (-\pi, \pi]$. It follows that:

- $|\det(A)| = |\sin \theta| \|v\| \|w\|$ is the area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$.
- $\det(A) = 0$ if $\vec{v}$ and $\vec{w}$ are parallel, i.e. $\theta = 0$, or $\theta = \pi$.
- $\det(A) > 0$ if $\theta \in (0, \pi)$.
- $\det(A) < 0$ if $\theta \in (-\pi, 0)$.
Example

Let

\[ \vec{v} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}. \]

Then

\[ \| \vec{v} \| = \sqrt{17}, \quad \| \vec{w} \| = \sqrt{13} \]

\[
\begin{align*}
\det \left( \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix} \right) &= \sin(0.737815\ldots) \sqrt{17} \sqrt{13} = 10 \\
\det \left( \begin{bmatrix} \vec{w} & \vec{v} \end{bmatrix} \right) &= \sin(-0.737815\ldots) \sqrt{17} \sqrt{13} = -10.
\end{align*}
\]
Available on Learning Glass videos:
2.4 — 1, 3, 9, 16, 17, 29, 31, 35
## Lecture – Book Roadmap

<table>
<thead>
<tr>
<th>Lecture</th>
<th>Book, [GS5–]</th>
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<td>§2.2</td>
</tr>
<tr>
<td>1.2</td>
<td>§1.1, §1.3, §2.1, §2.3</td>
</tr>
<tr>
<td>1.3</td>
<td>§1.1, §1.2, §1.3, §2.1, §2.3</td>
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<td>1.4</td>
<td>§1.1–§1.3, §2.1–§2.3</td>
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<tr>
<td>2.1</td>
<td>§8.1, §8.2*, §2.5*</td>
</tr>
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<td>§8.1, §8.2*, §4.2*, §4.4*</td>
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<tr>
<td>2.3</td>
<td>§2.4</td>
</tr>
<tr>
<td>2.4</td>
<td>§2.5</td>
</tr>
</tbody>
</table>

§2.5* (p.86–88) “Calculating $A^{-1}$ by Gauss-Jordan Elimination”
§4.2* (p.207) “Projection Onto a Line” – (p.210) end of “Example 2”
§4.4* Example 1, Example 3
§8.2* We will talk about “Basis” / “Bases” soon... don’t worry about those concepts... yet.
### Metacognitive Exercise — Thinking About Thinking & Learning

<table>
<thead>
<tr>
<th>I know / learned</th>
<th>Almost there</th>
<th>Huh?!?</th>
</tr>
</thead>
</table>

#### Right After Lecture

#### After Thinking / Office Hours / SI-session

#### After Reviewing for Midterm/Final
(2.4.1) Decide whether the matrix is invertible; if it is, find the inverse.

\[ A = \begin{bmatrix} 2 & 3 \\ 5 & 8 \end{bmatrix} \]

(2.4.3) Decide whether the matrix is invertible; if it is, find the inverse.

\[ A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \]
(2.4.9), (2.4.16)

(2.4.9) Decide whether the matrix is invertible; if it is, find the inverse.

\[ A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \]

(2.4.16) Decide whether the linear transformation is invertible; if it is, find the inverse transformation.

\[ y_1 = 3x_1 + 5x_2 \]
\[ y_2 = 5x_1 + 8x_2 \]
(2.4.17), (2.4.29)

(2.4.17) Decide whether the linear transformation is invertible; if it is, find the inverse transformation.

\[
\begin{align*}
    y_1 &= x_1 + 2x_2 \\
    y_2 &= 4x_1 + 8x_2
\end{align*}
\]

(2.4.29) For which values of the constant \( k \) is the following matrix invertible?

\[
A = \begin{bmatrix}
    1 & 1 & 1 \\
    1 & 2 & k \\
    1 & 4 & k^2
\end{bmatrix}
\]
For which values of the constants \( a, b, \) and \( c \) is the following matrix invertible?

\[
A = \begin{bmatrix}
0 & a & b \\
-a & 0 & c \\
-b & -c & 0
\end{bmatrix}
\]
(2.4.35)

a. Consider the upper triangular matrix

\[
A = \begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}
\]

For which values of \(a, b, c, d, e\), and \(f\) is \(A\) invertible?

b. More generally, when is an upper triangular matrix (of size \(n \times n\)) invertible?

c. If an upper triangular matrix is invertible, is its inverse also an upper triangular matrix?

d. Repeat questions b. and c. for lower triangular matrices.