Math 254: Introduction to Linear Algebra
Lecture Notes #3.1 — Image & Kernel of a Linear Transform

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Outline

1 Student Learning Objectives
   SLOs: Image & Kernel of a Linear Transform

2 Subspaces of \( \mathbb{R}^n \) and Their Dimensions
   Image & Kernel of a Linear Transformation

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SLOs 3.1

After this lecture you should:

- Be able to identify the image of a linear transformation (and its associated matrix) — \( \text{im}(A) \)
- Be able to identify the kernel of a linear transformation (and its associated matrix) — \( \text{ker}(A) \)
- Know what the span of a set of vectors is.
- Know when \( \text{ker}(A) = \{ \vec{0} \} \)? — and the implications [The Characteristics of Invertible Matrices]

Fair Warning

Things get quite “math-y” starting now.

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Definition (Image of a Function (Linear Transformation))

The image of a function consists of all the values the function takes in its target space [“range”]. If \( f : X \to Y \), then

\[
\text{image}(f) = \{ f(x) : x \in X \} = \{ b \in Y : b = f(x), \text{ for some } x \in X \}.
\]

Figure: \( X \) is the domain of \( f \); \( Y \) the target space of \( f \); and the shaded subset of \( Y \) is the image of \( f \).
Notational Hazard!

Note: “Range”

 Sometimes you see the term range in the literature; and depending on who is speaking (writing), it may refer to our target space, or what we call the image.

Example: \[ f(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} \] [NOT A LINEAR TRANSFORMATION]

Figure: The image of \( f(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} \) from \( \mathbb{R} \) to \( \mathbb{R}^2 \) consists of the unit circle centered at the origin. The function \( f \) is called the parametrization of the unit circle.

Example: \[ f(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ \cos(2t) \end{bmatrix} \] [NOT A LINEAR TRANSFORMATION]

Figure: The image of \( f(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ \cos(2t) \end{bmatrix} \) from \( \mathbb{R} \) to \( \mathbb{R}^3 \) consists of the figure above. (Here projected from 3D to 2D for your viewing pleasure!)
Consider \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) that projects a vector \( \vec{v} \) orthogonally onto the x-y-plane:

\[
T (\vec{x}) = A \vec{x} \quad \rightarrow \quad (k \vec{x}) = A k \vec{x} = k A \vec{x} = k T (\vec{x}).
\]

**Figure:** The image of \( T \) is the x-y-plane in \( \mathbb{R}^3 \), consisting of all vectors of the form

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} =
\begin{bmatrix}
x \\
y \\
0
\end{bmatrix}.
\]

Consider \( T (\vec{x}) := A \vec{x} \), with \( \vec{x} \in \mathbb{R}^2 \), and \( A \in \mathbb{R}^{2 \times 2} \), where

\[
A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}
\]

The image is described by

\[
\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 6 \end{bmatrix} = (x_1 + 3x_2) \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]

which is the line of all scalings of \( \begin{bmatrix} 1 \\ 2 \end{bmatrix} \).

**Figure:** The unit circle in domain space \( X = \mathbb{R}^2 \), and the image of the unit circle in target space \( Y = \mathbb{R}^2 \). Note: we can fill \( X = \mathbb{R}^2 \) with circles of radii \( r \in [0, \infty) \), so the image of \( T \) can be described by all scalings of the image of the unit circle; since \( T (k \vec{x}) = Ak \vec{x} = k A \vec{x} = k T (\vec{x}) \).

Consider \( T (\vec{x}) = A \vec{x} \), with \( \vec{x} \in \mathbb{R}^2 \), and \( A \in \mathbb{R}^{2 \times 2} \), where

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A = \begin{bmatrix} 1 & 3 \\ 6 & 2 \end{bmatrix}
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The image is described by

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\begin{bmatrix} 1 & 3 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix}
\]

which fills out all of \( \mathbb{R}^2 \) since \( \begin{bmatrix} 1 \\ 6 \end{bmatrix} \) and \( \begin{bmatrix} 3 \\ 2 \end{bmatrix} \) are not parallel.

**Figure:** The unit circle in domain space \( X = \mathbb{R}^2 \), and the image of the unit circle in target space \( Y = \mathbb{R}^2 \). Note: we can fill \( X = \mathbb{R}^2 \) with circles of radii \( r \in [0, \infty) \), so the image of \( T \) can be described by all scalings of the image of the unit circle; since \( T (k \vec{x}) = Ak \vec{x} = k A \vec{x} = k T (\vec{x}) \).
**The Span**

**Definition (The Span)**

Consider the vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \in \mathbb{R}^n \). The set of all linear combinations

\[
c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_m \vec{v}_m
\]

of the vectors is called their span:

\[
\text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m) = \{c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_m \vec{v}_m : c_1, c_2, \ldots, c_m \in \mathbb{R}\}.
\]

**Theorem (Image of a Linear Transformation)**

The image of a linear transformation \( T(\vec{x}) = A\vec{x} \) is the span of the column vectors of \( A \). We denote the image of \( T \) by \( \text{im}(T) \) or \( \text{im}(A) \).

**Notational Hazard (Language)**

Since \( \text{im}(A) \) is the span of the columns of \( A \), it is sometimes referred to as the **Column Space of** \( A \), denoted \( C(A) \) [GS5–3.1].

**Properties of \( \text{im}(T) \)**

**Theorem (Properties of the Image)**

The image of a linear transformation \( T : \mathbb{R}^m \rightarrow \mathbb{R}^n \) has the following properties:

a. The zero vector \( \vec{0} \) in \( \mathbb{R}^n \) is in the image of \( T \).

b. The image of \( T \) is closed under addition: if \( \vec{v}_1 \) and \( \vec{v}_2 \) are in the image of \( T \), then so is \( \vec{v}_1 + \vec{v}_2 \).

c. The image of \( T \) is closed under scalar multiplication: if \( \vec{v} \in \text{im}(T) \) and \( k \in \mathbb{R} \), then \( k\vec{v} \in \text{im}(T) \).

Proof in the supplemental slides.
Properties of the “Null Space”

The kernel (aka “null space”) of a linear transformation \( T(\vec{x}) = A\vec{x} \) from \( \mathbb{R}^m \) to \( \mathbb{R}^n \) consists of all zeros of the transformation; that is, the solutions of the equation \( T(\vec{x}) = A\vec{x} = \vec{0} \).

In other words, the kernel of \( T \) is the solution of the set of linear equations

\[ A\vec{x} = \vec{0} \]

We denote the kernel of \( T \) by \( \ker(T) \) or \( \ker(A) \).
Image & Kernel of a Linear Transformation

For the linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$,
- $\text{im}(T) = \{ T(\vec{x}) : \vec{x} \in \mathbb{R}^m \}$ is a subset of the target space $\mathbb{R}^n$ of $T$;
- $\text{ker}(T) = \{ \vec{x} \in \mathbb{R}^m : T(\vec{x}) = \vec{0} \}$ is a subset of the domain.

**Notational Hazard (Language)**

[GS5–3.2] uses the notation $N(A)$ for the null space (and [GS5–3.1] $C(A)$ for the image / column space). We will use $\text{ker}(A)$ and $\text{im}(A)$ exclusively.

**Example:** $\mathbb{R}^3 \to \mathbb{R}^3$

Consider, again, the linear transformation:

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \iff T(\vec{x}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

Clearly,

$$T \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \forall z \in \mathbb{R}.$$ 

Therefore,

$$\text{ker}(T) = \left\{ \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} : \forall z \in \mathbb{R} \right\}$$,
also $\text{im}(T) = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} : \forall x, y \in \mathbb{R} \right\}$.

**Example:** $\mathbb{R}^5 \to \mathbb{R}^4$

Find $\text{ker}(A)$

Consider $T(\vec{x}) = A\vec{x}$, where

$$A = \begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{bmatrix}$$

Let’s find the kernel (solve $A\vec{x} = \vec{0}$)

$$\begin{bmatrix} 1 & 2 & 2 & -5 & 6 \\ -1 & -2 & -1 & 1 & -1 \\ 4 & 8 & 5 & -8 & 9 \\ 3 & 6 & 1 & 5 & -7 \end{bmatrix}$$
The kernel is in \( \ker(A) \) if and only if \( A \) is invertible.

Given that the parameters, \( \{s, t, u\} \) are allowed to independently vary over \(( -\infty, \infty )\), we are interested in all combinations of the 3 vectors...

Using the previously defined concept of \( \text{span} \), we write

\[
\ker(T) = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}
\]

Properties of the Kernel

Theorem (Some Properties of the Kernel)

Consider the linear transform \( T : \mathbb{R}^m \rightarrow \mathbb{R}^n \),

a. The zero vector \( \vec{0} \) in \( \mathbb{R}^m \) is in \( \ker(T) \).

b. The kernel is closed under addition.

c. The kernel is closed under scalar multiplication.

The proofs for these properties are small modifications of the proofs of the analogous properties for the Image (see the Extended Notes)... and are left as an exercise.
### Characteristics of Invertible Matrices

**Summary: Invertible Matrices**

For an $n \times n$ matrix $A$, the following statements are equivalent; that is for a given $A$, they are either all true or all false:

1. $A$ is invertible ($\exists A^{-1}$)
2. The linear system $Ax = b$ has a unique solution $x$, $\forall b \in \mathbb{R}^n$
3. $\text{rref}(A) = I_n$
4. $\text{rank}(A) = n$
5. $\text{im}(A) = \mathbb{R}^n$
6. $\ker(A) = \{0\}$

We will add to this list throughout the semester.

### Lecture – Book Roadmap

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### Available on Learning Glass videos:

3.1 — 1, 7, 11, 14, 15, 17, 23, 24, 29, 39
(3.1.1), (3.1.7), (3.1.11)

(3.1.1) Find vectors that span the kernel of

\[ A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \]

(3.1.7) Find vectors that span the kernel of

\[ A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix} \]

(3.1.11) Find vectors that span the kernel of

\[ A = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 4 \end{bmatrix} \]

(3.1.14), (3.1.15), (3.1.17)

(3.1.14) Find vectors that span the image of

\[ A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \]

(3.1.15) Find vectors that span the image of

\[ A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \]

(3.1.17) Describe the image of the transformation \( T(\vec{x}) = A\vec{x} \) geometrically (e.g. as a line, a plane, etc. in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \)).

\[ A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \]

(3.1.23), (3.1.24)

(3.1.23) Describe the image and kernel of the transformation \( T(\vec{x}) = A\vec{x} \) geometrically, where

\[ T(\vec{x}) = \begin{cases} \text{Reflection about the line } y = \frac{x}{3} \text{ in } \mathbb{R}^2 \\ \{y = x/3\} \text{ in } \mathbb{R}^2 \end{cases} \].

(3.1.24) Describe the image and kernel of the transformation \( T(\vec{x}) = A\vec{x} \) geometrically, where

\[ T(\vec{x}) = \begin{cases} \text{Orthogonal projection onto the plane } \{x + 2y + 3z = 0\} \text{ in } \mathbb{R}^3 \\ \{x + 2y + 3z = 0\} \text{ in } \mathbb{R}^3 \end{cases} \].

(3.1.29), (3.1.39)

(3.1.29) Give an example of a function whose image is the unit sphere

\[ S^2 = \{x^2 + y^2 + z^2 = 1\} \text{ in } \mathbb{R}^3. \]

(3.1.39) Consider a square matrix \( A \):

a. What is the relationship among \( \ker(A) \) and \( \ker(A^2) \)? Are they necessarily equal? Is one of them necessarily contained in the other? More generally what can you say about \( \ker(A) \), \( \ker(A^2) \), \( \ker(A^3) \), ...?

b. What can you say about \( \text{im}(A) \), \( \text{im}(A^2) \), \( \text{im}(A^3) \), ...?