Outline

1 Student Learning Objectives
   - SLOs: Image & Kernel of a Linear Transform

2 Subspaces of $\mathbb{R}^n$ and Their Dimensions
   - Image & Kernel of a Linear Transformation

3 Suggested Problems
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   - Metacognitive Reflection
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After this lecture you should:

- Be able to identify the image of a linear transformation (and its associated matrix) — \( \text{im}(A) \)
- Be able to identify the kernel of a linear transformation (and its associated matrix) — \( \text{ker}(A) \)
- Know what the span of a set of vectors is.
- Know when \( \text{ker}(A) = \{ \vec{0} \} \)? — and the implications [The Characteristics of Invertible Matrices]

**Fair Warning**

Things get quite “math-y” starting now.
Definition (Image of a Function (Linear Transformation))

The *image* of a function consists of all the values the function takes in its target space [“*Range*”]. If \( f : X \to Y \), then

\[
\text{image}(f) = \{ f(x) : x \in X \}
\]

\[
= \{ b \in Y : b = f(x), \text{ for some } x \in X \}.
\]

**Figure:** \( X \) is the *domain* of \( f \); \( Y \) the *target space* of \( f \); and the shaded subset of \( Y \) is the *image* of \( f \).
Notational Hazard!

Note: “Range”
Sometimes you see the term **range** in the literature; and depending on who is speaking (writing), it may refer to our *target space*, or what we call the *image*. 

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Example $e^x : \mathbb{R} \rightarrow \mathbb{R}$

**Figure:** The image of $f(x) = e^x$ from $\mathbb{R}$ to $\mathbb{R}$ consists of $\mathbb{R}^+$ (all positive real numbers). Every positive number $b \in \mathbb{R}^+$ can be written as $b = e^{\ln(b)} = f(\ln(b))$. 
Example: $f(t) = \begin{bmatrix} \cos(t) & \sin(t) \end{bmatrix}^T$

[NOT A LINEAR TRANSFORMATION]

Figure: The image of $f(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}$ from $\mathbb{R}$ to $\mathbb{R}^2$ consists of the unit circle centered at the origin. The function $f$ is called the parametrization of the unit circle.
Example: \( f(t) = \begin{bmatrix} \cos(t) & \sin(t) & \cos(2t) \end{bmatrix}^T \) [NOT A LINEAR TRANSFORMATION]

**Figure:** The image of \( f(t) = \begin{bmatrix} \cos(t) \\ \sin(t) \\ \cos(2t) \end{bmatrix} \) from \( \mathbb{R} \) to \( \mathbb{R}^3 \) consists of the figure above. (Here projected from 3D to 2D for your viewing pleasure!)
Image of an Invertible Function

- If the function $f : X \to Y$ is invertible, then the image of $f$ is (all of) $Y$. \( \forall b \in Y \exists x \in X : b = f(x). \)
- In this case $x = f^{-1}(b)$:

\[
b = f(f^{-1}(b))
\]

See also Lecture 2.4.
Image of the Projection onto the $x$-$y$-Plane

Consider $T : \mathbb{R}^3 \to \mathbb{R}^3$ that projects a vector $\vec{v}$ orthogonally onto the $x$-$y$-plane:

$$
\begin{aligned}
T : \mathbb{R}^3 &\to \mathbb{R}^3 \\
\begin{bmatrix} x \\ y \\ z \end{bmatrix} &\mapsto \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}
\end{aligned}
$$

\[ T(\vec{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \]

**Figure:** The image of $T$ is the $x$-$y$-plane in $\mathbb{R}^3$, consisting of all vectors of the form

$$
\begin{bmatrix}
x \\ y \\ 0
\end{bmatrix}.
$$
Consider $T(\vec{x}) = A\vec{x}$, with $\vec{x} \in \mathbb{R}^2$, and $A \in \mathbb{R}^{2 \times 2}$, where

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}.$$  

The image is described by

$$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 6 \end{bmatrix} = (x_1 + 3x_2) \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

which is the line of all scalings of $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

**Figure:** The unit circle in domain space $X = \mathbb{R}^2$, and the image of the unit circle in target space $Y = \mathbb{R}^2$. Note: we can fill $X = \mathbb{R}^2$ with circles of radii $r \in [0, \infty)$, so the image of $T$ can be described by all scalings of the image of the unit circle; since $T(k\vec{x}) = Ak\vec{x} = kA\vec{x} = kT(\vec{x})$. 

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Consider \( T(\vec{x}) = A\vec{x} \), with \( \vec{x} \in \mathbb{R}^2 \), and \( A \in \mathbb{R}^{2 \times 2} \), where
\[
A = \begin{bmatrix} 1 & 3 \\ 6 & 2 \end{bmatrix}.
\]

The image is described by
\[
\begin{bmatrix} 1 & 3 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 6 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix}
\]
which fills out all of \( \mathbb{R}^2 \) since \( \begin{bmatrix} 1 \\ 6 \end{bmatrix} \) and \( \begin{bmatrix} 3 \\ 2 \end{bmatrix} \) are not parallel.

**Figure:** The unit circle in domain space \( X = \mathbb{R}^2 \), and the image of the unit circle in target space \( Y = \mathbb{R}^2 \). Note: we can fill \( X = \mathbb{R}^2 \) with circles of radii \( r \in [0, \infty) \), so the image of \( T \) can be described by all scalings of the image of the unit circle; since \( T(k\vec{x}) = Ak\vec{x} = kA\vec{x} = kT(\vec{x}) \).
Definition (The Span)
Consider the vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m \in \mathbb{R}^n \). The set of all linear combinations
\[
c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_m \vec{v}_m
\]
of the vectors is called their span:
\[
\text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m) = \{c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_m \vec{v}_m : c_1, c_2, \ldots, c_m \in \mathbb{R}\}.
\]
Theorem (Image of a Linear Transformation)

*The image of a linear transformation $T(\vec{x}) = A\vec{x}$ is the span of the column vectors of $A$. We denote the image of $T$ by $\text{im}(T)$ or $\text{im}(A)$.***

**WARNING** Notational Hazard (Language)

Since $\text{im}(A)$ is the span of the columns of $A$, it is sometimes referred to as the **Column Space of $A$**, denoted $C(A)$ [GS5–3.1].
Describing the Linear Transformation

The theorem pretty much proves itself; it follows directly from how we multiply vectors and matrices:

\[
T(\vec{x}) = A\vec{x} = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_m \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1 \vec{v}_1 + \cdots + x_m \vec{v}_m.
\]
Properties of $\text{im}(T)$

**Theorem (Properties of the Image)**

*The image of a linear transformation $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$ has the following properties:

a. The zero vector $\vec{0}$ in $\mathbb{R}^n$ is in the image of $T$.

b. The image of $T$ is **closed under addition**: if $\vec{v}_1$ and $\vec{v}_2$ are in the image of $T$, then so is $\vec{v}_1 + \vec{v}_2$.

c. The image of $T$ is **closed under scalar multiplication**: if $\vec{v} \in \text{im}(T)$ and $k \in \mathbb{R}$, then $k\vec{v} \in \text{im}(T)$.*

Proof in the supplemental slides.
Properties of the Image.

a. $\vec{0} = A\vec{0} = T(\vec{0})$.

b. $\exists \vec{w}_1, \vec{w}_2 \in \mathbb{R}^m: \vec{v}_1 = T(\vec{w}_1), \vec{v}_2 = T(\vec{w}_2)$. Then
\[ \vec{v}_1 + \vec{v}_2 = T(\vec{w}_1) + T(\vec{w}_2) \overset{\text{L.T.}}{=} T(\vec{w}_1 + \vec{w}_2) \Rightarrow (\vec{v}_1 + \vec{v}_2) \in \text{im}(T). \]

c. If $\vec{v} = T(\vec{w})$, then $k\vec{v} = kT(\vec{w}) \overset{\text{L.T.}}{=} T(k\vec{w})$. $\Rightarrow k\vec{v} \in \text{im}(T)$.  

[b.] + [c.] $\Rightarrow \text{im}(T)$ is closed under linear combinations.
Properties of the Image.

a. \( \vec{0} = A\vec{0} = T(\vec{0}) \).
   - This follows straight from how we compute matrix-vector products; given \( A \in \mathbb{R}^{n \times m} \), and \( T(\vec{x}) = A\vec{x} \), we immediately get \( A\vec{0}_m = \vec{0}_n \), where the subscript on the \( \vec{0} \)-vector indicates its number of components.

b. \( \exists \vec{w}_1, \vec{w}_2 \in \mathbb{R}^m: \vec{v}_1 = T(\vec{w}_1), \vec{v}_2 = T(\vec{w}_2) \).
   - Since \( \vec{v}_1 \) and \( \vec{v}_2 \) are in the image; there must exist ("\( \exists \)") vectors \( \vec{w}_1 \), and \( \vec{w}_2 \) so that \( \vec{v}_1 = T(\vec{w}_1), \vec{v}_2 = T(\vec{w}_2) \) \{some input must generate the output!\}

Then \( (\vec{v}_1 + \vec{v}_2) = T(\vec{w}_1) + T(\vec{w}_2) \) \( \overset{L.T.}{\Rightarrow} \) \( T(\vec{w}_1 + \vec{w}_2) \Rightarrow (\vec{v}_1 + \vec{v}_2) \in \text{im}(T) \).
   - First we write the vector we want to show is in the image \( (\vec{v}_1 + \vec{v}_2) \); then
   - ("\( = \)") we use the fact that each vector is in the image; followed by
   - ("\( L,T \)") the fact that \( T \) is a linear transformation; and we can conclude
   - ("\( \Rightarrow \)") that we wrote \( (\vec{v}_1 + \vec{v}_2) \) as the linear transformation of some vector \( \vec{w}^* = (\vec{w}_1 + \vec{w}_2) \), which makes \( \vec{v}^* = (\vec{v}_1 + \vec{v}_2) \) a member of the image.
Properties of the Image.

c. If $\vec{v} = T(\vec{w})$, then $k\vec{v} = kT(\vec{w}) \overset{L.T.}{=} T(k\vec{w})$. \(\Rightarrow k\vec{v} \in \text{im}(T)\).

- This is very similar to part ., given a vector $\vec{v}$ in the image; there must be a vector $\vec{w}$ in the domain, so that $\vec{v} = T(\vec{w})$.
- We want to show that $k\vec{v}$ is in the image; so we use $k\vec{v} = kT(\vec{w})$.
- then the fact that $T$ is a linear transformation: $kT(\vec{w}) = T(k\vec{w})$.
- and conclude as in part b.
The Kernel of a Linear Transformation

Definition (Kernel / Null Space)

The kernel (aka “null space”) of a linear transformation \( T(\vec{x}) = A\vec{x} \) from \( \mathbb{R}^m \) to \( \mathbb{R}^n \) consists of all zeros of the transformation; that is, the solutions of the equation \( T(\vec{x}) = A\vec{x} = \vec{0} \).

In other words, the kernel of \( T \) is the solution of the set of linear equations

\[ A\vec{x} = \vec{0} \]

We denote the kernel of \( T \) by \( \ker(T) \) or \( \ker(A) \).
For the linear transformation $T : \mathbb{R}^m \rightarrow \mathbb{R}^n$,

- $\text{im}(T) = \{ T(\vec{x}) : \vec{x} \in \mathbb{R}^m \}$ is a subset of the target space $\mathbb{R}^n$ of $T$;
- $\text{ker}(T) = \left\{ \vec{x} \in \mathbb{R}^m : T(\vec{x}) = \vec{0} \right\}$ is a subset of the domain.

**Notational Hazard (Language)**

[GS5–3.2] uses the notation $N(A)$ for the null space (and [GS5–3.1] $C(A)$ for the image / column space). We will use $\text{ker}(A)$ and $\text{im}(A)$ exclusively.
Example: \( \mathbb{R}^3 \rightarrow \mathbb{R}^3 \)

Consider, again, the linear transformation:

\[
T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad T(\vec{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}
\]

Clearly,

\[
T\left(\begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \forall z \in \mathbb{R}.
\]

Therefore,

\[
\ker(T) = \left\{ \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} : \forall z \in \mathbb{R} \right\}, \quad \text{also} \quad \text{im}(T) = \left\{ \begin{bmatrix} x \\ y \\ 0 \end{bmatrix} : \forall x, y \in \mathbb{R} \right\}.
\]
Example: \( \mathbb{R}^5 \rightarrow \mathbb{R}^4 \)

Consider \( T(\mathbf{x}) = A\mathbf{x} \), where

\[
A = \begin{bmatrix}
1 & 2 & 2 & -5 & 6 \\
-1 & -2 & -1 & 1 & -1 \\
4 & 8 & 5 & -8 & 9 \\
3 & 6 & 1 & 5 & -7
\end{bmatrix}
\]

Let’s find the kernel (solve \( A\mathbf{x} = \mathbf{0} \))

\[
\begin{bmatrix}
1 & 2 & 2 & -5 & 6 & 0 \\
-1 & -2 & -1 & 1 & -1 & 0 \\
4 & 8 & 5 & -8 & 9 & 0 \\
3 & 6 & 1 & 5 & -7 & 0
\end{bmatrix}
\]
Example: $\mathbb{R}^5 \rightarrow \mathbb{R}^4$

$$
\begin{bmatrix}
1 & 2 & 2 & -5 & 6 & | & 0 \\
-1 & -2 & -1 & 1 & -1 & | & 0 \\
4 & 8 & 5 & -8 & 9 & | & 0 \\
3 & 6 & 1 & 5 & -7 & | & 0
\end{bmatrix}
$$

$$
\begin{bmatrix}
1 & 2 & 2 & -5 & 6 & | & 0 \\
0 & 0 & 1 & -4 & 5 & | & 0 \\
0 & 0 & -3 & 12 & -15 & | & 0 \\
0 & 0 & -5 & 20 & -25 & | & 0
\end{bmatrix}
$$

$$
\begin{bmatrix}
1 & 2 & 0 & 3 & -4 & | & 0 \\
0 & 0 & 1 & -4 & 5 & | & 0 \\
0 & 0 & 0 & 0 & 0 & | & 0 \\
0 & 0 & 0 & 0 & 0 & | & 0
\end{bmatrix}
$$
Subspaces of $\mathbb{R}^n$ and Their Dimensions

Suggested Problems

Image & Kernel of a Linear Transformation

Example: $\mathbb{R}^5 \rightarrow \mathbb{R}^4$

$$\begin{bmatrix}
1 & 2 & 0 & 3 & -4 & | & 0 \\
0 & 0 & 1 & -4 & 5 & | & 0 \\
0 & 0 & 0 & 0 & 0 & | & 0 \\
0 & 0 & 0 & 0 & 0 & | & 0 \\
0 & 0 & 0 & 0 & 0 & | & 0 \\
\end{bmatrix}$$

rank$(A) = 2$

number-of-leading-variables $= 2$

number-of-free-variables $= 3$

Now, the equations

$$x_1 = -2x_2 - 3x_4 + 4x_5$$

$$x_3 = 4x_4 - 5x_5$$

describe the kernel. As usual we let $\{x_2 = s, x_4 = t, x_5 = u\}$, and write:

$$\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
\end{bmatrix} = \begin{bmatrix}
-2s - 3t + 4u \\
s \\
4t - 5u \\
t \\
u \\
\end{bmatrix} = s \begin{bmatrix}
-2 \\
1 \\
0 \\
0 \\
0 \\
\end{bmatrix} + t \begin{bmatrix}
3 \\
0 \\
4 \\
1 \\
0 \\
\end{bmatrix} + u \begin{bmatrix}
4 \\
0 \\
0 \\
0 \\
1 \\
\end{bmatrix}$$
Example: $\mathbb{R}^5 \to \mathbb{R}^4$

Given that the parameters, \{s, t, u\} are allowed to independently vary over \((-\infty, \infty)\), we are interested in all combinations of the 3 vectors...

Using the previously defined concept of span, we write

$$\ker(T) = \text{span} \left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$$
Theorem (Some Properties of the Kernel)

Consider the linear transform \( T : \mathbb{R}^m \to \mathbb{R}^n \),

a. The zero vector \( \vec{0} \) in \( \mathbb{R}^m \) is in \( \ker(T) \).

b. The kernel is closed under addition.

c. The kernel is closed under scalar multiplication.

The proofs for these properties are small modifications of the proofs of the analogous properties for the Image (see the Extended Notes)... and are left as an exercise.
When is \( \ker(A) = \{ \vec{0} \} \)?

**Theorem (When is \( \ker(A) = \{ \vec{0} \} \)?)**

- **a.** Consider an \( n \times m \) matrix \( A \). Then \( \ker(A) = \{ \vec{0} \} \) if and only if \( \text{rank}(A) = m \).

- **b.** Consider an \( n \times m \) matrix \( A \). If \( \ker(A) = \{ \vec{0} \} \), then \( m \leq n \). Equivalently, if \( m > n \), then there are non-zero vectors in the kernel of \( A \).

- **c.** For a square matrix \( A \), we have \( \ker(A) = \{ \vec{0} \} \) if and only if \( A \) is invertible.
Characteristics of Invertible Matrices

Summary: Invertible Matrices

For an $n \times n$ matrix $A$, the following statements are equivalent; that is for a given $A$, they are either all true or all false:

i. $A$ is invertible ($\exists A^{-1}$)

ii. The linear system $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$, $\forall \vec{b} \in \mathbb{R}^n$

iii. $\text{rref}(A) = I_n$

iv. $\text{rank}(A) = n$

v. $\text{im}(A) = \mathbb{R}^n$

vi. $\text{ker}(A) = \{\vec{0}\}$

We will add to this list throughout the semester.
Available on Learning Glass videos:

3.1 — 1, 7, 11, 14, 15, 17, 23, 24, 29, 39
## Lecture – Book Roadmap

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## Metacognitive Exercise — Thinking About Thinking & Learning

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(3.1.1), (3.1.7), (3.1.11)

**(3.1.1)** Find vectors that span the *kernel* of

\[ A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \]

**(3.1.7)** Find vectors that span the *kernel* of

\[ A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 3 & 2 & 1 \end{bmatrix} \]

**(3.1.11)** Find vectors that span the *kernel* of

\[ A = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 4 \end{bmatrix} \]
(3.1.14), (3.1.15), (3.1.17)

(3.1.14) Find vectors that span the image of

\[ A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \]

(3.1.15) Find vectors that span the image of

\[ A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix} \]

(3.1.17) Describe the image of the transformation \( T(\vec{x}) = A\vec{x} \) geometrically (e.g. as a line, a plane, etc. in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \)).

\[ A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \]
(3.1.23) Describe the *image* and *kernel* of the transformation $T(\vec{x}) = A\vec{x}$ geometrically, where

$$T(\vec{x}) = \begin{cases} \text{Reflection about the line } \{y = x/3\} \text{ in } \mathbb{R}^2 \\ \{y = x/3\} \text{ in } \mathbb{R}^2 \end{cases}.$$ 

(3.1.24) Describe the *image* and *kernel* of the transformation $T(\vec{x}) = A\vec{x}$ geometrically, where

$$T(\vec{x}) = \begin{cases} \text{Orthogonal projection onto the plane } \{x + 2y + 3z = 0\} \text{ in } \mathbb{R}^3 \end{cases}.$$
(3.1.29) Give an example of a function whose image is the unit sphere

\[ S^2 = \{ x^2 + y^2 + z^2 = 1 \} \text{ in } \mathbb{R}^3. \]

(3.1.39) Consider a square matrix \( A \):

a. What is the relationship among \( \ker(A) \) and \( \ker(A^2) \)? Are they necessarily equal?? Is one of them necessarily contained in the other? More generally what can you say about \( \ker(A), \ker(A^2), \ker(A^3), \ldots \)?

b. What can you say about \( \im(A), \im(A^2), \im(A^3), \ldots \)?