### Student Learning Objectives

**SLOs: Determinants**

**SLOs 6.1 Determinants**

After this lecture you should:

- Know that a Square Matrix has a Non-Zero Determinant if and only if it is Invertible
- Be familiar with the Connection between the determinant of a \((3 \times 3)\) matrix and the Cross Product (especially for Engineering / Physics students)
- Be able to compute the determinant using
  - Laplace (co-factor) Expansion Method ["TRADITIONAL" WAY].
  - Row Reductions,
- Know the Impact of Row Divisions/Swaps/Additions on the value of the Determinant
- Be familiar with computation of the Determinant of Products, Powers, Transposes, and Inverses of matrices: \(\det(AB)\), \(\det(A^k)\), \(\det(A^T)\), and \(\det(A^{-1})\)
- Forget about Cramer’s Rule: Don’t Use It! (Only for use in Joe Mahaffy’s Math 237/337 class)
Determinants — Modern Goals

We look at the minimum “essentials” / “executive summary” of determinants; mostly because the “traditional” view is prevalent, and you are likely to encounter small-matrix determinants in other classes.

Comment

For small matrices we will [Notes#7.2] use the determinant to find the characteristic polynomial, which will give us the eigenvalues. For non-small matrices, the minimal polynomial (which also reveals the eigenvalues) can be identified without the use of determinants.

Much of the material has been “banished” to the supplements, which make for an interesting read on a dark-and-stormy night.

The Determinant of a $(3 \times 3)$ Matrix

First, we “upsize” to the $(3 \times 3)$ case; let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = [\vec{u} \ \vec{v} \ \vec{w}]$$

The matrix is not invertible if the three column vectors are contained in a same plane $\iff$ they are linearly dependent.

In the $(3 \times 3)$ case it is popular to express the determinant in terms of the cross product...

The cross product can also be defined for $(7 \times 7)$ matrices [See Supplements]. For other $n$ we can define something “cross-product like” (e.g. the exterior, or “wedge”, product).
Slight Detour: The Cross Product in $\mathbb{R}^3$ 

**Definition (Cross Product in $\mathbb{R}^3$)**

The cross product $\mathbf{a} \times \mathbf{b}$ for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ is the vector in $\mathbb{R}^3$ with the following properties:

- $(\mathbf{a} \times \mathbf{b})$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.
- $||\mathbf{a} \times \mathbf{b}|| = |\sin(\theta)| ||\mathbf{a}|| ||\mathbf{b}||$; $\theta$ is the angle between $\mathbf{a}$ and $\mathbf{b}$; $\theta \in [0, \pi]$.
- The direction of $(\mathbf{a} \times \mathbf{b})$ follows the right-hand-rule.

**Theorem (Properties of the Cross Product)**

The following equations hold $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, and $\forall k \in \mathbb{R}$:

a. $\mathbf{v} \times \mathbf{w} = -\mathbf{w} \times \mathbf{v}$.

b. $(k \mathbf{v}) \times \mathbf{w} = k(\mathbf{v} \times \mathbf{w}) = \mathbf{v} \times (k \mathbf{w})$.

c. $\mathbf{v} \times (\mathbf{u} + \mathbf{w}) = \mathbf{v} \times \mathbf{u} + \mathbf{v} \times \mathbf{w}$.

d. $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ if and only if $\mathbf{v} \parallel \mathbf{w}$.

e. $\mathbf{v} \times \mathbf{v} = \mathbf{0}$.

f. $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$, $\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1$, $\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2$.

g. **the Jacobi Identity:** (This is the property that makes a cross product) $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = \mathbf{0}$

Note: Associativity is “missing.”

**Dot, Cross, and Outer Product**

- The Dot ("inner") Product, $\mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$
  - $\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b}$ is a scalar
  - $\mathbf{a} \cdot \mathbf{b} = \cos \theta ||\mathbf{a}|| ||\mathbf{b}||$

- The Cross Product, $\mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$
  - $\mathbf{a} \times \mathbf{b}$ is a vector $\perp \text{span} (\mathbf{a}, \mathbf{b})$
  - $||\mathbf{a} \times \mathbf{b}|| = \sin(\theta) ||\mathbf{a}|| ||\mathbf{b}||$

- The Outer Product, $\mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^{n \times n}$
  - $\mathbf{a} \mathbf{b}^T = \mathbf{M}$ is a matrix, $\mathbf{M} \in \mathbb{R}^{n \times n}$
  - $m_{ij} = a_i b_j$

$\theta$ is the angle between $\mathbf{a}$ and $\mathbf{b}$; $\theta \in [0, \pi]$.

$\exists$ Movie: CrossDot.mpeg
The (3 × 3) Determinant via the Cross Product

In the context of (3 × 3) matrices, we can compute
\[
\text{det} \left( \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix} \right) = \vec{u} \cdot (\vec{v} \times \vec{w})
\]

When \( \vec{u}, \vec{v}, \vec{w} \) are linearly dependent, then they “live” in the same plane; and \((\vec{v} \times \vec{w})\) is orthogonal to that plane, so the determinant is zero (as desired).

When \( \vec{u}, \vec{v}, \vec{w} \) are linearly independent, then \((\vec{v} \times \vec{w})\) is NOT orthogonal to \( \vec{u} \), and the determinant is non-zero.

Formula Overload!

So ponder:
\[
\text{det}(A) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} 
= \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} 
= a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{31}(a_{12}a_{23} - a_{13}a_{22})
\]

Committing the above to memory is not a good use of brain-power.

Suggested Problems

Introduction
The Road Forward...
More Properties and Interpretations...
Suggested Problems

The Road Forward
How to Compute \( \text{det}(A) \)

\( R^1 \times 1 \) and \( R^2 \times 2 \) Cases

\( R^1 \times 1 \): The matrix (and its associated linear transformation \( T : \mathbb{R}^1 \mapsto \mathbb{R}^1 \))
\[ A = \begin{bmatrix} a_{11} \end{bmatrix} \]
is invertible if and only if
\[ \text{det}(A) = a_{11} \neq 0. \]

\( R^2 \times 2 \): The matrix (and \( T : \mathbb{R}^2 \mapsto \mathbb{R}^2 \))
\[ A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \]
is invertible if and only if \([\text{Notes#2.4}]\)
\[ \text{det}(A) = a_{11}a_{22} - a_{12}a_{21} \neq 0. \]

Introduction
The Road Forward...
More Properties and Interpretations...
Suggested Problems

Determinants :: Executive Summary – Rationale
3 × 3 Determinant

Sarrus’ Rule
How to Compute \( \text{det}(A) \)

Only for \( \mathbb{R}^3 \times 3 \)

Theorem (Sarrus’ Rule, for \( \mathbb{R}^3 \times 3 \) only)
To find the determinant of a (3 × 3) matrix \( A \) to the right of \( A \), then multiply entries along the six diagonals shown:

Add the (blue / right-down) and subtract the (red / left-down) products, to get
\[ \text{det}(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \]

Unfortunately, Sarrus’ Rule does not generalize* to \((n \times n)\) matrices \((n \geq 4)\).

* This tends to lead to lost points on midterms and finals...
The Road Forward

How to Compute \( \det(A) \) \( \mathbb{R}^{n \times n}, n \geq 4 \)

Revisiting the (3 \times 3) case, we recall the result of Sarrus’ formula

\[ \det(A) = a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} - a_{13} a_{22} a_{31} \]

3 (of the 12) multiplications can be “saved” by writing

\[ \det(A) = a_{11}(a_{22} a_{33} - a_{23} a_{32}) + a_{21}(a_{13} a_{32} - a_{12} a_{33}) + a_{31}(a_{12} a_{23} - a_{13} a_{22}) \]

We can recognize this as

\[ \det(A) = a_{11} \det \left( \begin{array}{cc} * & a_{22} \\ a_{32} & a_{33} \end{array} \right) - a_{21} \det \left( \begin{array}{cc} * & a_{12} \\ a_{32} & a_{33} \end{array} \right) + a_{31} \det \left( \begin{array}{cc} * & a_{12} \\ a_{22} & a_{33} \end{array} \right) \]

\[ = a_{11} \det \left( \begin{array}{cc} * & a_{22} \\ a_{32} & a_{33} \end{array} \right) - a_{21} \det \left( \begin{array}{cc} * & a_{12} \\ a_{32} & a_{33} \end{array} \right) + a_{31} \det \left( \begin{array}{cc} * & a_{12} \\ a_{22} & a_{33} \end{array} \right) \]

We introduce a bit of notation and language...

Laplace (co-factor) Expansion \( O(n!) \) Work

Theorem (Laplace (co-factor) Expansion)

We can compute the determinant of an \( (n \times n) \) matrix \( A \) by Laplace expansion down any column, or along any row:
- Expansion down the \( j \)th column:
  \[ \det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}). \]
- Expansion along the \( i \)th row:
  \[ \det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}). \]

Matrix Minors

How to Compute \( \det(A) \) \( \mathbb{R}^{n \times n}, n \geq 4 \)

Definition (Minors)

For an \((n \times n)\) matrix \( A \), let \( A_{ij} \) be the matrix obtained by omitting the \( i \)th row, and the \( j \)th column of \( A \). The determinant of this \(((n-1) \times (n-1))\) matrix \( A_{ij} \) is called a minor of \( A \).

With this language, the determinant of the (3 \times 3) matrix \( A \):

\[ \det(A) = a_{11} \det(A_{11}) - a_{21} \det(A_{21}) + a_{31} \det(A_{31}). \]

This is known as the Laplace expansion, or co-factor expansion of \( \det(A) \) down the first column.

We generalize this common strategy...

Laplace (co-factor) Expansion

How to Compute \( \det(A) \) \( \mathbb{R}^{n \times n}, n \geq 4 \)

**Best Practice:** Select the row/column with the MOST zeros:

\[ \det \left( \begin{array}{ccc} 1 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 2 & 4 \end{array} \right) = (-1)^{1+2} \cdot 1 \cdot \det \left( \begin{array}{cc} 1 & 3 \\ 2 & 4 \end{array} \right) + (-1)^{2+3} \cdot 2 \cdot \det \left( \begin{array}{cc} 1 & 2 \\ 3 & 1 \end{array} \right) \]

Call Uncle Sarrus!

\((-1)^{(1+2)} \cdot 1 \cdot (1 \cdot 3 - 2 \cdot 4) + (-2)^{2+3} \cdot 2 \cdot (1 \cdot 1 - 2 \cdot 3) \]

\[= (-1)(6 + 36 - 8 - 4 - 18 = 24 = 30) \]

In general: An \((n \times n)\) determinant is computed using \( ((n-1) \times (n-1)) \) determinants... This means the work grows as \( n! \) factorial.
### Simplifying Cases

#### The 1st $\mathbb{R}^{4 \times 4}$ Determinant

Let's just go down the 1st column...

$$
\det \begin{pmatrix}
2 & 7 & 17 & 22 \\
3 & 8 & 18 & 23 \\
4 & 9 & 19 & 24 \\
5 & 10 & 20 & 25 \\
\end{pmatrix} = (+2) \det \begin{pmatrix}
8 & 18 & 23 \\
9 & 19 & 24 \\
10 & 20 & 25 \\
\end{pmatrix} + (+3) \det \begin{pmatrix}
7 & 17 & 22 \\
9 & 19 & 24 \\
10 & 20 & 25 \\
\end{pmatrix} + (+4) \det \begin{pmatrix}
7 & 17 & 22 \\
8 & 18 & 23 \\
9 & 19 & 24 \\
\end{pmatrix} + (+5) \det \begin{pmatrix}
7 & 17 & 22 \\
8 & 18 & 23 \\
9 & 19 & 24 \\
\end{pmatrix} = 0
$$

#### The 2nd $\mathbb{R}^{4 \times 4}$ Determinant

Let's grind through the 4 remaining $\mathbb{R}^{4 \times 4}$ determinants, we get 4 more zeros! That is massive amount of integer algebra to get 0 zeros...

$$
\begin{align*}
\det \begin{pmatrix}
1 & 6 & 16 & 21 \\
2 & 7 & 17 & 22 \\
3 & 8 & 18 & 23 \\
4 & 9 & 19 & 24 \\
\end{pmatrix} &= 0, \\
\det \begin{pmatrix}
1 & 6 & 16 & 21 \\
2 & 7 & 17 & 22 \\
3 & 8 & 18 & 23 \\
5 & 10 & 20 & 25 \\
\end{pmatrix} &= 0,
\end{align*}
\begin{align*}
\det \begin{pmatrix}
1 & 6 & 16 & 21 \\
2 & 7 & 17 & 22 \\
3 & 8 & 18 & 23 \\
5 & 10 & 20 & 25 \\
\end{pmatrix} &= 0,
\end{align*}
\begin{align*}
\det \begin{pmatrix}
1 & 6 & 16 & 21 \\
2 & 7 & 17 & 22 \\
3 & 8 & 18 & 23 \\
4 & 9 & 19 & 24 \\
\end{pmatrix} &= 0.
\end{align*}
$$

Therefore, $\det \begin{pmatrix}
1 & 6 & 16 & 21 \\
2 & 7 & 17 & 22 \\
3 & 8 & 18 & 23 \\
4 & 9 & 19 & 24 \\
5 & 10 & 20 & 25 \\
\end{pmatrix} = 0$.

#### 2nd Generation Problem

$$
\det \begin{pmatrix}
2 & 7 & 17 & 22 \\
3 & 8 & 18 & 23 \\
4 & 9 & 19 & 24 \\
5 & 10 & 20 & 25 \\
\end{pmatrix} = (+2) \det \begin{pmatrix}
8 & 18 & 23 \\
9 & 19 & 24 \\
10 & 20 & 25 \\
\end{pmatrix} + (+3) \det \begin{pmatrix}
7 & 17 & 22 \\
9 & 19 & 24 \\
10 & 20 & 25 \\
\end{pmatrix} + (+4) \det \begin{pmatrix}
7 & 17 & 22 \\
8 & 18 & 23 \\
9 & 19 & 24 \\
\end{pmatrix} + (+5) \det \begin{pmatrix}
7 & 17 & 22 \\
8 & 18 & 23 \\
9 & 19 & 24 \\
\end{pmatrix} = 0
$$

#### 2nd Generation Problem

$$
\det \begin{pmatrix}
2 & 7 & 17 & 22 \\
3 & 8 & 18 & 23 \\
4 & 9 & 19 & 24 \\
5 & 10 & 20 & 25 \\
\end{pmatrix} = (+2) \det \begin{pmatrix}
8 & 18 & 23 \\
9 & 19 & 24 \\
10 & 20 & 25 \\
\end{pmatrix} + (+3) \det \begin{pmatrix}
7 & 17 & 22 \\
9 & 19 & 24 \\
10 & 20 & 25 \\
\end{pmatrix} + (+4) \det \begin{pmatrix}
7 & 17 & 22 \\
8 & 18 & 23 \\
9 & 19 & 24 \\
\end{pmatrix} + (+5) \det \begin{pmatrix}
7 & 17 & 22 \\
8 & 18 & 23 \\
9 & 19 & 24 \\
\end{pmatrix} = 0
$$

#### Triangular Matrices

**Example (Upper/Lower Triangular Matrix)**

**Let**

$$A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
0 & a_{22} & a_{23} \\
0 & 0 & a_{33} \\
\end{bmatrix}, \quad B = \begin{bmatrix}
b_{11} & 0 & 0 \\
b_{21} & b_{22} & 0 \\
b_{31} & b_{32} & b_{33} \\
\end{bmatrix}.$$

Then $\det(A) = a_{11}a_{22}a_{33}$, and $\det(B) = b_{11}b_{22}b_{33}$

**Theorem (Determinant of a Triangular Matrix)**

The determinant of a triangular matrix is the product of the diagonal entries of the matrix.

In particular, the determinant of a diagonal matrix is the product of its diagonal entries.
Row-Reductions and Determinants

Our three fundamental row-operations are:

1. **Row division**: Dividing a row by a non-zero scalar $k$.
2. **Row swap**: Swapping two rows.
3. **Row addition**: Adding (subtracting) a multiple of one row to another.

It is natural to ask how these operations change the value of the determinant.

Row-Reductions and Determinants: The $(2 \times 2)$ Case

First, consider the $(2 \times 2)$ case $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with $\det(A) = ad - bc$:

1. **Row division**: If $B = \begin{bmatrix} a/k & b/k \\ c & d \end{bmatrix}$, then
   \[
   \det(B) = \frac{ad}{k} - \frac{bc}{k} = \det(A)/k.
   \]
   \[
   \sim \text{Scaling of the Determinant}
   \]

2. **Row swap**: If $B = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$, then
   \[
   \det(B) = cb - da = -\det(A).
   \]
   \[
   \sim \text{Sign Change of the Determinant}
   \]

3. **Row addition**: If $B = \begin{bmatrix} a + kc & b + kd \\ c & d \end{bmatrix}$, then
   \[
   \det(B) = (a + kc)d - (b + kd)c = (ad - bc) + k(cd - dc) = \det(A).
   \]
   \[
   \sim \text{No Change to the Determinant}
   \]

Properties for all $(n \times n)$-matrices

**Theorem (Elementary Row Operations and Determinants)**

a. If $B$ is obtained from $A$ by dividing a row of $A$ by a scalar $k$, then
   \[
   \det(B) = \frac{1}{k}\det(A)
   \]

b. If $B$ is obtained from $A$ by a row swap, then
   \[
   \det(B) = -\det(A)
   \]

c. If $B$ is obtained from $A$ by adding a multiple of a row of $A$ to another row, then
   \[
   \det(B) = \det(A)
   \]

Analogous results hold for elementary column operations.

Relating $\det(\text{rref}(A))$ and $\det(A)$

Now, if we in the process of computing the reduced-row-echelon-form of a matrix $A$

- count the number of row-swaps: $s$, and
- keep track of scalar divisions $k_1, \ldots, k_r$.

then:

\[
\det(\text{rref}(A)) = (-1)^s \frac{1}{k_1 k_2 \ldots k_r} \det(A),
\]

or

\[
\det(A) = (-1)^s k_1 k_2 \ldots k_r \det(\text{rref}(A))
\]

This can save a lot of work for large matrices; Computing $\text{rref}(A)$ requires \( \sim \mathcal{O}(n^3) \) work, which is a grows a lot slower than $\mathcal{O}(n!)$.
Gauss-Jordan Elimination and the Determinant

If instead of computing \( \det(\text{rref}(A)) \), we perform elementary row operations on \( A \) to transform it into some matrix \( B \), where \( \det(B) \) is easy to compute; the same rules apply; if we performed \( s \) row swaps, and scaled rows by the factors \( k_1, \ldots, k_r \), then

\[
\det(A) = (-1)^s k_1 k_2 \ldots k_r \det(B)
\]

Transforming \( A \) into upper triangular form \( U \) is a popular choice, since

\[
\det(U) = \prod_{k=1}^{n} u_{kk}.
\]

This approach saves about half the work vs. computing \( \det(\text{rref}(A)) \). \textbf{This is the clever thing to do!}

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**Suggested Problems**

\[ R1 \times 1; R2 \times 2; R3 \times 3; Rn \times n, n \ldots \]

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**More Properties and Interpretations...**

\[ R^5 \times 5 \text{ Number Crunching Bonanza — Redux!} \]

Once again, we compute:

\[
\begin{vmatrix}
1 & 6 & 11 & 16 & 21 \\
2 & 7 & 12 & 17 & 22 \\
3 & 8 & 13 & 18 & 23 \\
4 & 9 & 14 & 19 & 24 \\
5 & 10 & 15 & 20 & 25
\end{vmatrix}
\]

First, we eliminate the first column:

\[
\begin{pmatrix}
1 & 6 & 11 & 16 & 21 \\
0 & -5 & -10 & -15 & -20 \\
0 & -10 & -20 & -30 & -40 \\
0 & -15 & -30 & -45 & -60 \\
0 & -20 & -40 & -60 & -80
\end{pmatrix}
\]

We divide the rows by \((-5), (-10), (-15), (-20); \) and eliminate down the 2nd column...

\[
\begin{pmatrix}
1 & 6 & 11 & 16 & 21 \\
0 & 1 & 2 & 3 & 4 \\
0 & 1 & 2 & 3 & 4 & \sim & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 3 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 3 & 4 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 2 & 3 & 4 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\[ (s = 0) \] since we did not swap any rows; the row-division factors are back as multiplications; and \( \det(U) \) is just the product of the diagonal entries

\[
\det(U) = 1 \cdot 1 \cdot 0 \cdot 0 \cdot 0 = 0
\]

and again it follows that \( \det(A) = 0. \)
Determinant of the Transpose

**Theorem (Determinant of the Transpose)**

*If $A$ is a square matrix, then*

$$\det(A^T) = \det(A).$$

- This means that any property expressed in terms of columns/rows is also true for rows/columns;

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Linearity of the Determinant in the Columns (Rows)

**Theorem (Linearity of the Determinant in the Columns)**

*Consider fixed column vectors*

$$\vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_i + \vec{x}, \vec{v}_{i+1}, \ldots, \vec{v}_n \in \mathbb{R}^n.$$

*Then the function $T : \mathbb{R}^n \mapsto \mathbb{R}$ defined by*

$$T(\vec{x}) = \det(\begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_{i-1} & \vec{x} & \vec{v}_{i+1} & \cdots & \vec{v}_n \end{bmatrix})$$

*is a linear transformation.*

We can convince ourselves that the theorem is indeed true...

---

Invertibility and Determinant

*If $A$ is invertible, then $rref(A) = I$, so that*

$$\det(rref(A)) = \det(I_n) = 1,$$ 

and

$$\det(A) = (-1)^s k_1 k_2 \ldots k_r \neq 0.$$

*If $A$ is non-invertible, then the last row of $rref(A)$ is all zeros, and by linearity $\det(rref(A)) = 0$; so that $\det(A) = 0$.**

**Theorem (Invertibility and Determinant)**

*A square matrix $A$ is invertible if and only if $\det(A) \neq 0$.*

---

Yes! we can assign a number $\det(A)$ to any square matrix $A$, such that $A$ is invertible if and only if $\det(A) \neq 0$!
Determinants of Products and Powers

**Theorem (Determinants of Products and Powers)**

*If $A$ and $B$ are $(n \times n)$ matrices, and $m \in \mathbb{Z}^+$ is a positive integer, then*

- a. $\det(AB) = (\det(A))(\det(B))$, and
- b. $\det(A^m) = (\det(A))^m$.

**Proof**

[Proof in the supplements]

Determinants of Similar Matrices

**Example (Similar Matrices)**

Consider two similar matrices $A$, $B$; where $S$ is an invertible matrix so that

$$AS = SB.$$  

The previous theorems then implies that

$$\det(A) \det(S) = \det(S) \det(B),$$

so that $\det(A) = \det(B)$.

Cramer’s Rule

**Theorem (Cramer’s Rule)**

*Consider the linear system*

$$A\vec{x} = \vec{b}$$

*where $A$ is an invertible $(n \times n)$ system. The components $x_i$ of the solution vector $\vec{x}$ are*

$$x_i = \frac{\det(A_{\bar{i},i})}{\det(A)},$$

*where $A_{\bar{i},i}$ is the matrix obtained by replacing the $i^{th}$ column of $A$ by $\vec{b}$.*

**Proof:**

[Short: relies on fundamental properties/definitions].

Since $I_n = A^{-1}A$, $1 = \det(I_n) = \det(A^{-1})\det(A)$. 

The Second Worst Idea in Linear Algebra

**Peter’s Postulate**

Solving linear systems using Cramer’s Rule is a BAD IDEA. — We need to compute $(n + 1)$ determinants of size $(n \times n)$. 

Peter Blomgren (blomgren@sdsu.edu) 6.1. Determinants :: Executive Summary — (40/100)
The Worst Idea in Linear Algebra...

Rewind (Definition: Minors)
For an \((n \times n)\) matrix \(A\), let \(A_{ij}\) be the matrix obtained by omitting the \(i^{th}\) row, and \(j^{th}\) column of \(A\). The determinant of the \(((n - 1) \times (n - 1))\) matrix \(A_{ij}\) is called a minor of \(A\).

Now:
Definition (The Classical Adjoint)
The classical adjoint \(M = \text{adj}(A)\) of an invertible \((n \times n)\) matrix, is the matrix whose \(j^{th}\) entry \(m_{ij} = (-1)^{i+j} \det(A_{ji})\). Yes, we have to compute \(n^2\) \[((n - 1) \times (n - 1))\) determinants to build the adjoint! With “only” one more \((n \times n)\) determinant, we can express the inverse:

\[
A^{-1} = \frac{1}{\det(A)} \text{adj}(A).
\]

Examples...

1. Compute \(\det(A)\), where \(A \in \mathbb{R}^{n \times n}\); \(n = 1, 2, 3\); or \(n > 3\); with “structure” \{lots of zeros, or some other greatly simplifying characteristic\}
   - using row-reductions
   - using Laplace co-factor expansion
   - using appropriate “short-cut rule(s)”

2. Given matrices \(A, B\) compute \(\det(A), \det(B), \det(AB), \det(A^{4006}), \det(B^{-1}), \det(A^T B^{-1} A B^T)\); etc...

3. Given the value of \(\det(A)\) is the matrix invertible?

4. Given a matrix \(A\), with \(\det(A)\); how does \(\det(A)\) change under row/column operations
### Metacognitive Reflection

**Thinking About Thinking & Learning**

- **I know / learned**
- **Almost there**
- **Huh?!?**

### Outline

6. Supplemental Material :: Problems
   - Metacognitive Reflection
   - Problem Statements 6.1
   - Problem Statements 6.2
   - Problem Statements 6.3

7. Supplemental Material — [Focus :: Math]
   - The 7-Dimensional Cross Product
   - Determinants of Products and Powers
   - Application: Ordinary Differential Equations (ODEs)

8. Supplemental Material — General
   - Geometric Interpretation: \(n\)-Dimensional Expansion Factor
   - \((n \times n)\) Determinant (“Pattern Method”)
   - Pattern Example: Determinant of Sparse \((4 \times 4)\) Matrix
   - Block Matrices (“Pattern” Approach)
   - Computational Feasibility of the Determinant

### Problem Statements 6.1

(6.1.1), (6.1.5)

**6.1.1** Is the matrix

\[ A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \]

invertible?

**6.1.5** Is the matrix

\[ A = \begin{bmatrix} 2 & 5 & 7 \\ 0 & 11 & 7 \\ 0 & 0 & 5 \end{bmatrix} \]

invertible?

### Problem Statements 6.2

(6.1.11), (6.1.13)

**6.1.11** For which values of \(k\) is the matrix

\[ A = \begin{bmatrix} k & 2 \\ 3 & 4 \end{bmatrix} \]

invertible?

**6.1.13** For which values of \(k\) is the matrix

\[ A = \begin{bmatrix} k & 3 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 4 \end{bmatrix} \]

invertible?
(6.1.23), (6.1.25)

(6.1.23) Use the determinant to find out for which values of \( \lambda \) the matrix \((A - \lambda I_n)\)
\[
A = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}
\]
is invertible?

(6.1.25) Use the determinant to find out for which values of \( \lambda \) the matrix \((A - \lambda I_n)\)
\[
A = \begin{bmatrix} 4 & 2 \\ 4 & 6 \end{bmatrix}
\]
is invertible?

(6.1.27), (6.1.31)

(6.1.27) Use the determinant to find out for which values of \( \lambda \) the matrix \((A - \lambda I_n)\)
\[
A = \begin{bmatrix} 2 & 0 & 0 \\ 5 & 3 & 0 \\ 7 & 6 & 4 \end{bmatrix}
\]
is invertible?

(6.1.31) Find the determinant of
\[
A = \begin{bmatrix} 1 & 9 & 8 & 7 \\ 0 & 2 & 9 & 6 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 4 \end{bmatrix}
\]

(6.2.1), (6.2.5)

(6.2.1) Use Gaussian Elimination (Row Reductions) to find the determinant of the matrix
\[
A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 2 & 2 & 5 \end{bmatrix}
\]

(6.2.5) Use Gaussian Elimination (Row Reductions) to find the determinant of the matrix
\[
A = \begin{bmatrix} 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 4 \\ 1 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \end{bmatrix}
\]
Use Gaussian Elimination (Row Reductions) to find the determinants of the matrices

\[
(6.2.7) \quad A = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 2 \\
0 & 1 & 2 & 3 \\
1 & 2 & 3 & 4
\end{bmatrix}, \quad (6.2.9) \quad B = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 \\
1 & 1 & 3 & 3 \\
1 & 1 & 1 & 5
\end{bmatrix}.
\]

Find the area of the triangle with corners in \((6.3.3)\) and \((6.3.1)\).

Consider a \((4 \times 4)\) matrix \(A\) with rows \(\vec{v}_1, \vec{v}_2, \vec{v}_3, \text{ and } \vec{v}_4\). If \(\det(A) = 8\), what is:

\[
(6.2.11) \quad \det \begin{bmatrix}
\vec{v}_1 \\
\vec{v}_2 \\
-9 \vec{v}_3 \\
\vec{v}_4
\end{bmatrix}, \quad (6.2.12) \quad \det \begin{bmatrix}
\vec{v}_4 \\
\vec{v}_3 \\
\vec{v}_2 \\
\vec{v}_1
\end{bmatrix},
\]

\[
(6.2.13) \quad \det \begin{bmatrix}
\vec{v}_3 \\
\vec{v}_2 \\
\vec{v}_1 \\
\vec{v}_4
\end{bmatrix}, \text{ and } (6.2.15) \quad \det \begin{bmatrix}
\vec{v}_1 + \vec{v}_2 \\
\vec{v}_1 + \vec{v}_2 + \vec{v}_3 \\
\vec{v}_1 + \vec{v}_2 + \vec{v}_3 + \vec{v}_4
\end{bmatrix}.
\]

The tetrahedron defined by three vectors \(\vec{v}_1, \vec{v}_2, \text{ and } \vec{v}_3 \in \mathbb{R}^3\) is the set of all vectors of the form \(c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3\), where \(c_i \geq 0\), and \(c_1 + c_2 + c_3 \leq 1\). Explain why the volume is one sixth of the volume of the parallelepiped defined by \(\vec{v}_1, \vec{v}_2, \text{ and } \vec{v}_3\).
(6.3.7), (6.3.9), (6.3.11)

(6.3.7) Find the area of the region with corners in \([-7, -5, 3, 5]\).

(6.3.9) If \(\vec{v}_1\) and \(\vec{v}_2\) are linearly independent vectors in \(\mathbb{R}^2\), what is the relationship between \(\det([\vec{v}_1, \vec{v}_2])\) and \(\det([\vec{v}_1, \vec{v}_2]^T)\), where \(\vec{v}_2^T\) is the component of \(\vec{v}_2\) orthogonal to \(\vec{v}_1\).

(6.3.11) Consider a linear transformation \(T(x) = Ax\) from \(\mathbb{R}^2\) to \(\mathbb{R}^2\). Suppose for two vectors \(\vec{v}_1\) and \(\vec{v}_2\) in \(\mathbb{R}^2\) we have \(T(\vec{v}_1) = 3\vec{v}_1\) and \(T(\vec{v}_2) = 4\vec{v}_2\). What can you say about \(\det(A)\)? Explain in detail.

---

(6.3.13), (6.3.19), (6.3.20)

(6.3.13) Find the 2-volume (aka “area”) of the 2-parallelepiped (parallelogram) defined by the two vectors

\[
\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \quad \text{and} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 4 \end{bmatrix}.
\]

(6.3.19) A basis \((\vec{v}_1, \vec{v}_2, \vec{v}_3)\) of \(\mathbb{R}^3\) is called positively oriented if \(\vec{v}_1\) encloses an acute angle with \((\vec{v}_2 \times \vec{v}_3)\). Illustrate with a sketch. Show that the basis is positively oriented if-and-only-if \(\det([\vec{v}_1, \vec{v}_2, \vec{v}_3])\) is positive.

(6.3.20) We say that a linear transformation \(T: \mathbb{R}^3 \rightarrow \mathbb{R}^3\) preserves orientation if it transforms any positively oriented basis into another positively oriented basis. Explain why a linear transformation \(T(x) = Ax\) preserves orientation if-and-only-if \(\det(A) > 0\).

---

(6.3.21)

(6.3.21) Arguing geometrically, determine whether the following orthogonal transformations from \(\mathbb{R}^3\) to \(\mathbb{R}^3\) preserve, or reverse orientation:

a. Reflection about a plane.
b. Reflection about a line.
c. Reflection about the origin.

---

The 7D cross product is a bilinear operation on vectors in seven-dimensional Euclidean space. For any \(\vec{a}, \vec{b} \in \mathbb{R}^7\) it assigns a vector \(\vec{v} = (\vec{a} \times \vec{b}), \vec{v} \in \mathbb{R}^7\).

Like the 3D cross product, the 7D cross product is anticommutative and \((\vec{a} \times \vec{b}) \perp \text{span}(\vec{a}, \vec{b})\).

Unlike in three dimensions, it does not satisfy the Jacobi identity, and while the 3D cross product is unique up to a sign, there are many seven-dimensional cross products.

The 7D cross product has the same relationship to the octonions as the three-dimensional product does to the quaternions (a number system that extends to complex numbers).
The 7-Dimensional Cross Product

Crazy-Math-Interlude

[FOCUS :: Math]

This is not meant to be useful... it's just for “fun!”

The 7D cross product is one way of generalizing the cross product to other than 3D, and it is the only other non-trivial bilinear product of two vectors that is
(i) vector-valued,
(ii) anticommutative, and
(iii) orthogonal.

In other dimensions there are vector-valued products of three or more vectors that satisfy these conditions, and binary products with bivector results.

Jacobi Identity: a property of a binary operation which describes how the order of evaluation (the placement of parentheses in a multiple product) affects the result of the operation:

A binary operation $\circ$ on a set $S$ possessing a binary operation $+$ with an additive identity denoted by $0$ satisfies the Jacobi identity if:

$$a \circ (b \circ c) + b \circ (c \circ a) + c \circ (a \circ b) = 0 \quad \forall a, b, c \in S.$$

By contrast, for operations with the associative property, any order of evaluation gives the same result (parentheses in a multiple product are not needed).

Named after the German mathematician Carl Gustav Jakob Jacobi.

Real Numbers: basis $\{1\}$, with multiplication table:

$$\begin{array}{c|c|c}
\times & 1 & 1 \\
\hline
1 & 1 & 1 \\
\end{array}$$

Complex Numbers: basis $\{1, i\}$, with multiplication table:

$$\begin{array}{c|c|c}
\times & 1 & i \\
\hline
1 & 1 & i \\
1 & i & -1 \\
\end{array}$$

Quaternions: basis $\{1, i, j, k\}$, with multiplication table:

$$\begin{array}{c|c|c|c|c}
\times & 1 & i & j & k \\
\hline
1 & 1 & i & j & k \\
i & i & -1 & k & -j \\
j & j & -k & -1 & i \\
k & k & j & -i & -1 \\
\end{array}$$
**The 7-Dimensional Cross Product**

### Crazy-Math-Interlude

**Octonions**: basis — \{1, i, j, k, ℓ, m, n, o\}, with multiplication table:

\[
\begin{array}{cccccccc}
1 & i & j & k & ℓ & m & n & o \\
i & 1 & i & j & k & ℓ & m & n \\
j & i & 1 & j & k & ℓ & m & n \\
k & j & 1 & j & k & ℓ & m & n \\
ℓ & ℓ & m & ℓ & m & n & ℓ & m \\
m & ℓ & m & n & ℓ & m & n & ℓ \\
n & ℓ & m & n & ℓ & m & n & ℓ \\
o & ℓ & m & n & ℓ & m & n & ℓ \\
\end{array}
\]

**3D Cross Product**: basis — \{i, j, k\}, with multiplication table:

\[
\begin{array}{ccc}
i & j & k \\
j & 0 & \mathbf{e}_2 \\
k & \mathbf{e}_2 & -\mathbf{e}_1 \\
\end{array}
\]

This is not meant to be useful... it's just for "fun!"

---

**Determinants of Products and Powers**

Theorem (Determinants of Products and Powers)

If \(A\) and \(B\) are \((n \times n)\) matrices, and \(m \in \mathbb{Z}^+\) is a positive integer, then

a. \(\det(AB) = (\det(A)) (\det(B))\), and

b. \(\det(A^m) = (\det(A))^m\).
(a.)

(i) First we assume $A$ is invertible: The row-operations required to transform $A$ to $I_n$, applied to the augmented system $[A \mid AB]$ gives:

$$\text{rref} ([A \mid AB]) = [I_n \mid I_nB] = [I_n \mid B]$$

i.e. they are equivalent to multiplying both sides of the augmentation by $A^{-1}$.

Keeping track of the row-swaps, and row divisions (scalings) $k_1, \ldots, k_r$ required to transform $A$ into its RREF-form, we get

$$\det(A) = (-1)^s k_1 k_2 \ldots k_r,$$

and

$$\det(AB) = (-1)^s k_1 k_2 \ldots k_r \det(B) = (\det(A)) \det(B).$$

(b.)

Apply part (a.) $(m-1)$ times to get:

$$\det(A^m) = \det(AA^{m-1}) = \det(A)\det(A^{m-1}) = \cdots = (\det(A))^m$$
ODEs: Variation of Parameters  

**Linear Algebra Connection!**

[FOCUS :: Math]

This gives two linear algebraic equations in $u_1'$ and $u_2'$

$$
u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0$$

$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t),$$

or in matrix form

$$
\begin{bmatrix}
    y_1(t) & y_2(t) \\
    y_1'(t) & y_2'(t)
\end{bmatrix}
\begin{bmatrix}
    u_1'(t) \\
    u_2'(t)
\end{bmatrix} =
\begin{bmatrix}
    0 \\
    g(t)
\end{bmatrix}
$$

As long as the columns of $A(t)$ are linearly independent, we get solutions

[Notes #2.4]:

$$
\begin{bmatrix}
    u_1'(t) \\
    u_2'(t)
\end{bmatrix} = \frac{1}{\det(A(t))}
\begin{bmatrix}
    y_2'(t) - y_2(t) \\
    -y_1'(t) & y_1(t)
\end{bmatrix}
\begin{bmatrix}
    0 \\
    g(t)
\end{bmatrix}
$$

ODEs: Variation of Parameters  

**Theorem (Variation of Parameters)**

Consider the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t),$$

If the functions $p(t), q(t),$ and $g(t)$ are continuous on an open interval $I \subset \mathbb{R}^n,$ and if $y_1$ and $y_2$ form a fundamental set of solutions of the homogeneous equation. Then a particular solution of the nonhomogeneous problem is

$$y_p(t) = -y_1(t) \int_{t_0}^{t} \frac{y_2(s)g(s)}{W[y_1, y_2](s)} ds + y_2(t) \int_{t_0}^{t} \frac{y_1(s)g(s)}{W[y_1, y_2](s)} ds,$$

where $t_0 \in I$. The general solution is

$$y(t) = c_1y_1(t) + c_2y_2(t) + y_p(t).$$

ODEs: Variation of Parameters  

**Determinant / “Wronskian”**

In the “ODE Universe,” $\det(A(t)) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$ is usually referred to as the “Wronskian”, and sometimes denoted $W[y_1, y_2](t)$.

We can now integrate

$$u_1(t) = -\int_{t}^{t} \frac{y_2(\tau)g(\tau)}{\det(A(\tau))} d\tau + C_1, \quad u_2(t) = \int_{t}^{t} \frac{y_1(\tau)g(\tau)}{\det(A(\tau))} d\tau + C_2,$$

and the general solution is given by

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t);$$

where $u_1(t)$ and $u_2(t)$ be be given explicitly, or in integral form.
The Determinant as Expansion Factor

**Focus :: Math**

**The Area**

Which gives us

\[
\frac{\sin(2t)}{2} = \cos(t) \sin(t), \quad \cos(2t) = 2 \cos^2(t) - 1
\]

So that

\[
\frac{-3 \sin(t) \sin(2t)}{2}, \quad \frac{3 \sin(t) \cos(2t)}{2}
\]

Which gives us

\[
u_1(t) = -\sin^3(t) + C_1, \quad u_2(t) = \frac{3}{2} \cos(t) - \cos^3(t) + C_2
\]

Consider a linear transformation \( T : \mathbb{R}^2 \mapsto \mathbb{R}^2 \). We have discussed how such a transform impacts lengths, and angles. For a transform \( \mathbb{R}^2 \mapsto \mathbb{R}^2 \) it also makes sense to think about the 2-volume (aka “The Area”); and for \( \mathbb{R}^n \mapsto \mathbb{R}^n \) we can discuss the \( m \)-Volume(s).
The Determinant as Expansion Factor

We start in the $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ case, and let the “input area” $(\Omega)$ be the unit square (with area 1), described by the two vectors $\vec{e}_1$ and $\vec{e}_2$.

The “output area” $(T(\Omega))$, is then described by $A\vec{e}_1 = \vec{v}_1$, and $A\vec{e}_2 = \vec{v}_2$, i.e. the parallelepiped spanned by the columns of $A$; here the area is $|\det(A)|$.

The expansion factor is

$$\frac{\text{area of } T(\Omega)}{\text{area of } \Omega} = \frac{|\det(A)|}{1} = |\det(A)|.$$  

The Determinant as Expansion Factor

Consider a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(\vec{x}) = A\vec{x}$. Then $|\det(A)|$ is the expansion factor of $T$ on parallelograms $\Omega$.

Likewise, for linear transformations $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by $T(\vec{x}) = A\vec{x}$, $|\det(A)|$ is the expansion factor of $T$ on $n$-parallelepipeds:

$$V(A\vec{v}_1, \ldots, A\vec{v}_n) = |\det(A)| V(\vec{v}_1, \ldots, \vec{v}_n),$$

for all vectors $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n$.

The Determinant as Expansion Factor

If the input parallelepiped is described by two vectors $\vec{w}_1$, and $\vec{w}_2$, then the original area is $|\det(B)|$, where $B = [\vec{w}_1 \quad \vec{w}_2]$.

The “output area” $(T(\Omega))$, is then described by $A\vec{w}_1 = \vec{v}_1$, and $A\vec{w}_2 = \vec{v}_2$; so the area of $T(\Omega)$ is given by

$$|\det([A\vec{w}_1 \quad A\vec{w}_2])| = |\det(AB)| = |\det(A)||\det(B)|.$$

The expansion factor is

$$\frac{\text{area of } T(\Omega)}{\text{area of } \Omega} = \frac{|\det(A)||\det(B)|}{|\det(B)|} = |\det(A)|.$$
6.1. Determinants :: Executive Summary

A “Fun” Definition of Determinants

The following discussion introduces the
● “Pattern,” or
● “Combinatorial” ← USEFUL FOR INTERNET SEARCHING...
definition of matrix determinants.

The Determinant of an \((n \times n)\) Matrix

For an \((n \times n)\) matrix \(A:\)

- Let a pattern, \(P\), of \(A\) be a subset (vector) containing \(n\) entries \(a_{ij}\) selected from the matrix \(A\) so that we have exactly one entry from each row and column. (There are \(n(n-1)(n-2)\cdots 1 = n! \) “n-factorial” such patterns.)
- Let \(\text{prod}(P)\) be the product of all entries in a pattern \(P\).
- An inversion in a pattern occurs when an entry is to the right, and above another; e.g. if the pattern contains \(a_{12}\), and \(a_{31}\) we have an inversion.
- Let \(\text{sgn}(P)\) be the signature of the pattern, defined as \((-1)^{\text{#inversions of } P}.\)

What the @@@#@%&***???!!! Let’s go back and apply this to the
\((3 \times 3)\) case.

Patterns, Inversions, Products, and Signatures in the \((3 \times 3)\) case...

We have established that

\[
\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}
\]

\[
\begin{array}{ccc}
\text{Pattern Example: Determinant of Sparse } (4 \times 4) \text{ Matrix} \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{Block Matrices ("Pattern Approach")} \\
\end{array}
\]

Computational Feasibility of the Determinant

\[
\begin{array}{ccc}
\text{Geometric Interpretation: n-Dimensional Expansion Factor} \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{(n \times n) Determinant ("Pattern Method")} \\
\end{array}
\]

Supplemental Material — General

Note: Fun with Factorials

They Grow FAST!

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</tr>
</tbody>
</table>

Bottom line: there are lots of patterns for (not very) large matrices.
The Determinant of an \((n \times n)\) Matrix

We can now define the determinant of an \((n \times n)\) matrix using its associated patterns (and the products, inversions and signatures of those patterns):

\[
\text{Definition (The Determinant of an } (n \times n) \text{ Matrix)}
\]

\[\det(A) = \sum_{\text{All } n! \text{ patterns of } A} \text{sgn}(P) \prod(P)\]

Determinant of a Sparse \((4 \times 4)\) Matrix

The combinatorial “pattern” approach can be computationally efficient when there are lots of zeros in the matrix; patterns with a 0 will not contribute to the value of the determinant, so we can skip exploring those patterns...

We consider computing the determinant of

\[
A = \begin{bmatrix}
0 & 0 & 1 & 2 \\
0 & 1 & 1 & 3 \\
1 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 \\
\end{bmatrix}
\]

Patterns including \(a_{31} = 1\):

\[
\begin{bmatrix}
x & 0 & 1 & 2 \\
x & 1 & 1 & 3 \\
1 & x & x & x \\
x & 1 & 0 & 0 \\
\end{bmatrix}
\]

Is This Computationally Useful? .... Collecting Examples

All of what we have stated is true, but a big chunk of it is not really computationally practical for general matrices.

However, for matrices with lots of zeros (“Sparse Matrices”) this approach can be time-saving, since any pattern containing a zero will not contribute to the determinant.

Further, most properties (which we will discuss) of determinants can be traced back to the combinatorial definition; and some computational strategies leverage those properties.

Determinant of a Sparse \((4 \times 4)\) Matrix

[continued] Patterns including \(a_{31} = 1\):

\[
\begin{bmatrix}
x & x & 1 & 2 \\
x & 1 & x & x \\
1 & x & x & x \\
x & x & 0 & 0 \\
\end{bmatrix}
\]

\(P = \{a_{31}, a_{22}, \ldots\}\)

We notice that for the left pattern, the next non-zero choice is \(a_{13} = 1\), but that leaves \(a_{44} = 0\) for the final selection; so this is a zero-end pattern. The right pattern can be competed in two ways:

\[
\begin{bmatrix}
x & x & 1 & x \\
x & x & x & x \\
1 & x & 3 & x \\
x & 1 & x & x \\
\end{bmatrix}
\]

\(P = \{a_{31}, a_{42}, a_{13}, a_{24}\}\)

or

\[
\begin{bmatrix}
x & x & x & 1 \\
x & x & x & x \\
1 & x & x & x \\
x & 1 & x & x \\
\end{bmatrix}
\]

\(P = \{a_{31}, a_{42}, a_{23}, a_{14}\}\)
Determinant of a Sparse \((4 \times 4)\) Matrix

[continued] Patterns including \(a_{31} = 1\):

\[
\begin{bmatrix}
1 & 1 & x & x \\
1 & x & x & x \\
x & x & 1 & x \\
x & x & x & 1
\end{bmatrix}

\]

or

\[
\begin{bmatrix}
1 & 1 & x & x \\
1 & x & x & x \\
x & x & 1 & x \\
x & x & x & 1
\end{bmatrix}

\]

The pattern \(P = \{a_{31}, a_{42}, a_{13}, a_{24}\}\) has 4 inversions, so \(\text{sgn}(P) = 1\), and \(\text{prod}(P) = 1 \cdot 1 \cdot 1 \cdot 3 = 3\). Contribution to \(\det(A)\) :: \((+3)\).

The pattern \(P = \{a_{31}, a_{42}, a_{23}, a_{14}\}\) has 5 inversions, so \(\text{sgn}(P) = -1\), and \(\text{prod}(P) = 1 \cdot 1 \cdot 1 \cdot 2 = 2\). Contribution to \(\det(A)\) :: \((-2)\).

In summary, the determinant of

\[
A = \begin{bmatrix}
0 & 0 & 1 & 2 \\
0 & 1 & 1 & 3 \\
1 & 0 & 0 & 0 \\
2 & 1 & 0 & 0
\end{bmatrix}
\]

is given by

\[
\det(A) = \text{sgn}(P_1) \text{prod}(P_1) + \text{sgn}(P_2) \text{prod}(P_2) = (+1)(+3) + (-1)(+2) = 1
\]

where

\[
P_1 = \{a_{31}, a_{42}, a_{13}, a_{24}\} = \{1, 1, 1, 3\},
\]

\[
P_2 = \{a_{31}, a_{42}, a_{23}, a_{14}\} = \{1, 1, 1, 2\}
\]

are the only 2 (out of 24) patterns with non-zero contributions.

Next, we have to consider Patterns including \(a_{41} = 2\):

\[
\begin{bmatrix}
x & x & 1 & 2 \\
x & 1 & x & x \\
x & x & 0 & 0 \\
x & x & x & x
\end{bmatrix}
\]

There's only 1 non-zero choice in the 2nd column; and then in third: which forces us to pick a zero in the 4th column...

Since the pattern \(P = \{a_{41}, a_{22}, a_{13}, a_{34}\}\) contains a zero, we have \(\text{prod}(P) = 2 \cdot 1 \cdot 1 \cdot 0 = 0\), it does not contribute to \(\det(A)\).
Block Matrices

Now, 
\[
\det(M) = a_{11}a_{22}c_{11}c_{22} - a_{11}a_{22}c_{12}c_{21} - a_{12}a_{21}c_{11}c_{22} + a_{12}a_{21}c_{12}c_{21}
\]
\[
= (a_{11}a_{22} - a_{12}a_{21})(c_{11}c_{22} - c_{12}c_{21})
\]
\[
= \det(A)\det(C).
\]

Computational Feasibility of the Determinant

Say you are faced with computing the determinant of a \((32 \times 32)\) matrix (not very large by modern standards).

Using the “pattern” (or Laplace co-factor) method, such a computation would require \(31 \cdot 32! \approx 8.16 \cdot 10^{36}\) multiplications.

Now say you have access to an “exascale computer” — which can perform \(10^{18}\) operations / second; then your computation would only take about \(10^{18}\) seconds... how long is that?

- 100 seconds/minute \(\sim 10^{16}\) minutes
- 100 minutes/hour \(\sim 10^{14}\) hours
- 100 hours/day \(\sim 10^{12}\) days
- 1000 days/year \(\sim 10^9\) years.
- so... only about 7% of the age of the universe.

Perfect midterm question, yeah?!?

As of November 2019, according to https://www.top500.org/, the fastest supercomputer in the world:

<table>
<thead>
<tr>
<th>Site</th>
<th>Oak Ridge National Laboratory</th>
</tr>
</thead>
<tbody>
<tr>
<td>System</td>
<td>Summit - IBM Power System AC922, IBM POWER9 22C 3.07GHz, NVIDIA Volta GV100, Dual-rail Mellanox EDR Infiniband</td>
</tr>
<tr>
<td>Cores</td>
<td>2,414,592</td>
</tr>
<tr>
<td>Rmax</td>
<td>148,600.0 TFlop/s</td>
</tr>
<tr>
<td>Rpeak</td>
<td>200,794.9 TFlop/s (0.201 \times 10^{18} Flop/s)</td>
</tr>
<tr>
<td>Power</td>
<td>10,096 kW</td>
</tr>
</tbody>
</table>