Student Learning Objectives

- SLOs: Determinants

Introduction

- Detour: The Cross Product in \( \mathbb{R}^3 \)
  - 3 \( \times \) 3 Determinant
  - \( n \times n \) Determinant

Moving Along...

- Useful Examples and Results

Suggested Problems

- Suggested Problems 6.1
- Lecture–Book Roadmap

Determinants: Introduction

The matrix
\[
A = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix}
\]
is invertible if and only if [Notes 2.4]
\[
\det(A) = ad - bc \neq 0.
\]

The quantity \( ad - bc \) is called the determinant of the matrix \( A \).

It is natural to ask: Can we assign a number \( \det(A) \) to any square matrix \( A \), such that \( A \) is invertible if-and-only-if \( \det(A) \neq 0 \)?

To our euphoric joy, the answer is “yes!”
The Determinant of a 3 × 3 Matrix

First, we "upsiz[e]" to the 3 × 3 case:

Let

\[ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = [\bar{u} \ \bar{v} \ \bar{w}] \]

The matrix is not invertible if the three column vectors are contained in a same plane \( \iff \) they are linearly dependent.

In the 3 × 3 case it is popular to express the determinant in terms of the cross product... In the \( n \times n \) \((n \geq 4)\) setting, we tend to use the determinant to define a generalized cross product. The generalization is only called a cross product when \( n = 7 \).

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**Slight Detour: The Cross Product in \( \mathbb{R}^3 \)**

**Definition (Cross Product in \( \mathbb{R}^3 \))**

The cross product \( \vec{a} \times \vec{b} \) for \( \vec{a}, \vec{b} \in \mathbb{R}^3 \) is the vector in \( \mathbb{R}^3 \) with the following properties:

- \( \vec{a} \times \vec{b} \) is orthogonal to both \( \vec{a} \) and \( \vec{b} \).
- \( ||\vec{a} \times \vec{b}|| = \sin(\theta) ||\vec{a}|| ||\vec{b}|| \); \( \theta \) is the angle between \( \vec{a} \) and \( \vec{b} \), \( \theta \in [0, \pi] \).
- The direction of \( \vec{a} \times \vec{b} \) follows the right-hand-rule.

**Properties**

- The Dot ("inner") Product, \( \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \)
  \( \vec{a} \cdot \vec{b} = \vec{a}^T \vec{b} \) is a scalar
  \( \vec{a} \cdot \vec{b} = \cos \theta ||\vec{a}|| ||\vec{b}|| \)
- The Cross Product, \( \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \)
  \( \vec{a} \times \vec{b} \) is a vector \( \perp \text{ span } (\vec{a}, \vec{b}) \)
  \( ||\vec{a} \times \vec{b}|| = \sin(\theta) ||\vec{a}|| ||\vec{b}|| \)
- The Outer Product, \( \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times n} \)
  \( \vec{a} \vec{b}^T = M \) is a matrix, \( M \in \mathbb{R}^{n \times n} \)
  \( m_{ij} = a_i b_j \)

\( \theta \) is the angle between \( \vec{a} \) and \( \vec{b} \); \( \theta \in [0, \pi] \).

\( \exists \) Movie: Cross Dot.mpeg

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**Theorem (Properties of the Cross Product)**

The following equations hold \( \forall \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3 \), and \( \forall k \in \mathbb{R} \):

\[ \begin{align*}
  a. \quad & \vec{v} \times \vec{w} = -\vec{w} \times \vec{v}. \\
  b. \quad & (k\vec{v}) \times \vec{w} = k(\vec{v} \times \vec{w}) = \vec{v} \times (k\vec{w}) \\
  c. \quad & \vec{v} \times (\vec{u} + \vec{w}) = \vec{v} \times \vec{u} + \vec{v} \times \vec{w} \\
  d. \quad & \vec{v} \times \vec{w} = \vec{0} \iff \vec{v} \parallel \vec{w} \\
  e. \quad & \vec{v} \times \vec{v} = \vec{0} \\
  f. \quad & \vec{e}_1 \times \vec{e}_2 = \vec{e}_3, \quad \vec{e}_2 \times \vec{e}_3 = \vec{e}_1, \quad \vec{e}_3 \times \vec{e}_1 = \vec{e}_2. 
\end{align*} \]

(right-hand-rule)
Slight Detour: The Cross Product in $\mathbb{R}^3$

Theorem (The Cross Product in Components)

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \times \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{bmatrix}$$

OK, back to determinants...

Formula Overload!

So ponder:

$$\det(A) = \vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} a_{11} \\ a_{21} \\ a_{31} \end{vmatrix} \cdot \left( \begin{vmatrix} a_{12} \\ a_{22} \\ a_{32} \end{vmatrix} \times \begin{vmatrix} a_{13} \\ a_{23} \\ a_{33} \end{vmatrix} \right) = a_{11} \begin{vmatrix} a_{22} a_{33} - a_{23} a_{32} \\ a_{21} a_{33} - a_{23} a_{31} \\ a_{21} a_{32} - a_{22} a_{31} \end{vmatrix} = a_{11} a_{22} a_{33} + a_{11} a_{23} a_{32} + a_{12} a_{23} a_{31} - a_{12} a_{21} a_{33} - a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}$$

Let nightmares ensue!

The 3 × 3 Determinant via the Cross Product

In the context of 3 × 3 matrices, we can compute

$$\det \left( \begin{bmatrix} \vec{u} \\ \vec{v} \\ \vec{w} \end{bmatrix} \right) = \vec{u} \cdot (\vec{v} \times \vec{w})$$

When $\vec{u}, \vec{v}, \vec{w}$ are linearly dependent, then they “live” in the same plane; and $\vec{v} \times \vec{w}$ is orthogonal to that plane, so the determinant is zero (as desired).

When $\vec{u}, \vec{v}, \vec{w}$ are linearly independent, then $\vec{v} \times \vec{w}$ is NOT orthogonal to $\vec{u}$, and the determinant is non-zero.

Sarrus’ Rule

How to Compute $\det(A)$

Only for $\mathbb{R}^{3 \times 3}$

Theorem (Sarrus’ Rule)

To find the determinant of a 3 × 3 matrix $A$, write the first 2 columns of $A$ to the right of $A$, then multiply entries along the six diagonals shown:

$$\det(A) = a_{11} a_{22} a_{33} + a_{11} a_{23} a_{32} + a_{12} a_{23} a_{31} - a_{12} a_{21} a_{33} - a_{13} a_{21} a_{32} - a_{13} a_{22} a_{31}$$

Unfortunately, Sarrus’ Rule does not generalize* to $n \times n$ matrices ($n > 3$).

* This tends to lead to lost points on midterms and finals...
The Determinant of an $n \times n$ Matrix

For an $n \times n$ matrix $A$:
- Let a pattern, $P$, of $A$ be a subset (vector) containing $n$ entries $a_{ij}$ selected from the matrix $A$ so that we have exactly one entry from each row and column. (There are $n!(n-1)(n-2)\cdots 1 = n!$ “$n$-factorial” such patterns.)
- Let $\text{prod}(P)$ be the product of all entries in a pattern $P$.
- An inversion in a pattern occurs when an entry is to the right, and above another; e.g. if the pattern contains $a_{12}$, and $a_{31}$ we have an inversion.
- Let $\text{sgn}(P)$ be the signature of the pattern, defined as $(-1)^{\# \text{inversions of $P$}}$.

What the @@@#%&***???!!! Let’s go back and apply this to the $3 \times 3$ case.

Patterns, Inversions, Products, and Signatures in the $3 \times 3$ case...

We have established that

\[ \det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \]

No inversions.

\[ \begin{array}{ccc}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{array} \]

2 Inversions:
- $a_{31} : a_{12}, a_{31} : a_{23}$

\[ \begin{array}{ccc}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{array} \]

2 Inversions:
- $a_{21} : a_{13}, a_{32} : a_{13}$

\[ \begin{array}{ccc}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{array} \]

1 Inversion:
- $a_{32} : a_{13}$

\[ \begin{array}{ccc}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{array} \]

1 Inversion:
- $a_{31} : a_{12}$

\[ \begin{array}{ccc}
    a_{11} & a_{12} & a_{13} \\
    a_{21} & a_{22} & a_{23} \\
    a_{31} & a_{32} & a_{33}
\end{array} \]

3 Inversions:
- $a_{31} : a_{12}, a_{31} : a_{13}, a_{31} : a_{23}$

Suggested Problems

Moving Along...

Note: Fun with Factorials

The Determinant of an $n \times n$ Matrix

We can now define the determinant of an $n \times n$ matrix using its associated patterns (and the products, inversions and signatures of those patterns):

**Definition (The Determinant of an $n \times n$ Matrix)**

\[ \det(A) = \sum \text{sgn}(P) \text{prod}(P) \]

All $n!$ patterns of $A$
All of what we have stated is true, but a big chunk of it is not really computationally practical for general matrices.

In the next lecture we will look at more properties of the determinant, and introduce structured (practical) ways to compute the determinant.

But, first let’s collect some useful examples...

For which values of $\lambda$ is the matrix

$$A = \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & -1 \\ 1 & 1 & -\lambda \end{bmatrix}$$

invertible?

**Solution:** We compute

$$\det(A) = -\lambda^3 + \lambda = \lambda(\lambda + 1)(\lambda - 1),$$

which shows that $A$ is invertible except when $\lambda \in \{-1, 0, 1\}.$

Consider

$$\det(A) = \det\left(\begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}\right) = \vec{u} \cdot (\vec{v} \times \vec{w})$$

and

$$\det(B) = \det\left(\begin{bmatrix} \vec{u} & \vec{w} & \vec{v} \end{bmatrix}\right) = \vec{u} \cdot (\vec{w} \times \vec{v})$$

Since the cross-product is anti-commutative, we must have

$$\det(A) = -\det(B).$$

This is true in general, swapping rows or columns will flip the sign of the determinant.

The determinant of a triangular matrix is the product of the diagonal entries of the matrix.

In particular, the determinant of a diagonal matrix is the product of its diagonal entries.
Suggested Problems 6.1

Theorem (Determinant of a Block Matrix)

If $A$ and $C$ are square matrices (not necessarily of the same size), and $B$ and $0$ are matrices of appropriate size, then

$$
\det \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \det(A) \det(C), \quad \det \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} = \det(A) \det(C).
$$

**WARNING**

The formula

$$
\det \begin{bmatrix} A & B & C \\ B & D & C \end{bmatrix} = \det(A) \det(D) - \det(B) \det(C)
$$

does NOT always hold.

See [https://en.wikipedia.org/wiki/Determinant#Block_matrices](https://en.wikipedia.org/wiki/Determinant#Block_matrices)

Available on Learning Glass videos:

6.1 — 1, 5, 11, 13, 23, 25, 27, 31, 43, 45
Lecture – Book Roadmap

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<td>§5.1, §5.2, §5.3</td>
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<td>6.2</td>
<td>§5.1, §5.2, §5.3</td>
</tr>
<tr>
<td>6.3</td>
<td>§5.1, §5.2, §5.3</td>
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* Strang does not talk about the combinatorial (pattern) definition of the determinant.