1 Student Learning Objectives
   • SLOs: Determinants

2 Introduction
   • Detour: The Cross Product in $\mathbb{R}^3$
   • $3 \times 3$ Determinant
   • $n \times n$ Determinant

3 Moving Along…
   • Useful Examples and Results

4 Suggested Problems
   • Suggested Problems 6.1
   • Lecture–Book Roadmap
After this lecture you should:

- Know that a Square Matrix has a Non-Zero Determinant if and only if it is Invertible
- Be familiar with the Connection between the determinant of a $3 \times 3$ matrix and the Cross Product (especially for Engineering / Physics students)
- Be familiar with Definition of the Determinant using the Combinatorial “Pattern” approach, and be able to use this definition to compute determinants of sparse matrices (i.e. matrices that have LOTS of zero-entries)
Determinants: Introduction

The matrix

\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

is invertible \textbf{if and only if} [Notes 2.4]

\[ \det(A) = ad - bc \neq 0. \]

The quantity \((ad - bc)\) is called the \textit{determinant} of the matrix \(A\).

It is natural to ask: \textit{Can we assign a number} \(\det(A)\) \textit{to any square matrix} \(A\), \textit{such that} \(A\) \textit{is invertible if-and-only-if} \(\det(A) \neq 0\)?

To our euphoric joy, the answer is “yes!”
The Determinant of a $3 \times 3$ Matrix

First, we “upsize” to the $3 \times 3$ case:

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = [\vec{u} \ \vec{v} \ \vec{w}]$$

The matrix is not invertible if the three column vectors are contained in a same plane $\iff$ they are linearly dependent.

In the $3 \times 3$ case it is popular to express the determinant in terms of the cross product... In the $n \times n$ ($n \geq 4$) setting, we tend to use the determinant to define a generalized cross product. The generalization is only called a cross product when $n = 7$. 
Definition (Cross Product in $\mathbb{R}^3$)

The cross product $\vec{a} \times \vec{b}$ for $\vec{a}, \vec{b} \in \mathbb{R}^3$ is the vector in $\mathbb{R}^3$ with the following properties:

- $\vec{a} \times \vec{b}$ is orthogonal to both $\vec{a}$ and $\vec{b}$.
- $\|\vec{a} \times \vec{b}\| = \sin(\theta) \|\vec{a}\| \|\vec{b}\|; \theta$ is the angle between $\vec{a}$ and $\vec{b}$; $\theta \in [0, \pi]$.\(^*\)
- The direction of $\vec{a} \times \vec{b}$ follows the right-hand-rule

**Figure:** The right-hand-rule.

**Figure:** *This means that $\|\vec{a} \times \vec{b}\|$ is the area of the parallelogram spanned by $\vec{a}$ and $\vec{b}$.\(^*\)
Theorem (Properties of the Cross Product)

The following equations hold $\forall \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3$, and $\forall k \in \mathbb{R}$:

- **a.** $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$. *(anti-commutative)*
- **b.** $(k\vec{v}) \times \vec{w} = k(\vec{v} \times \vec{w}) = \vec{v} \times (k\vec{w})$.
- **c.** $\vec{v} \times (\vec{u} + \vec{w}) = \vec{v} \times \vec{u} + \vec{v} \times \vec{w}$
- **d.** $\vec{v} \times \vec{w} = \vec{0}$ *if and only if* $\vec{v} \parallel \vec{w}$.
- **e.** $\vec{v} \times \vec{v} = \vec{0}$.
- **f.** $\vec{e}_1 \times \vec{e}_2 = \vec{e}_3$, $\vec{e}_2 \times \vec{e}_3 = \vec{e}_1$, $\vec{e}_3 \times \vec{e}_1 = \vec{e}_2$. *(right-hand-rule)*
The Dot ("inner") Product, $\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$
- $\vec{a} \cdot \vec{b} = \vec{a}^T \vec{b}$ is a scalar
- $\vec{a} \cdot \vec{b} = \cos \theta \|\vec{a}\| \|\vec{b}\|$

The Cross Product, $\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$
- $\vec{a} \times \vec{b}$ is a vector $\perp \text{span}(\vec{a}, \vec{b})$
- $\|\vec{a} \times \vec{b}\| = \sin(\theta) \|\vec{a}\| \|\vec{b}\|$

The Outer Product, $\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$
- $\vec{a} \vec{b}^T = M$ is a matrix, $M \in \mathbb{R}^{n \times n}$
- $m_{ij} = a_i b_j$

$\theta$ is the angle between $\vec{a}$ and $\vec{b}$; $\theta \in [0, \pi]$.

∃ Movie: Cross Dot.mpeg
Theorem (The Cross Product in Components)

\[
\begin{pmatrix}
v_1 \\
v_2 \\
v_3
\end{pmatrix}
\times
\begin{pmatrix}
w_1 \\
w_2 \\
w_3
\end{pmatrix}
=
\begin{pmatrix}
v_2 w_3 - v_3 w_2 \\
v_3 w_1 - v_1 w_3 \\
v_1 w_2 - v_2 w_1
\end{pmatrix}
\]

OK, back to determinants...
The 3 × 3 Determinant via the Cross Product

In the context of 3 × 3 matrices, we can compute

\[
\text{det} \left( \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix} \right) = \vec{u} \cdot (\vec{v} \times \vec{w})
\]

When \( \vec{u}, \vec{v}, \vec{w} \) are linearly dependent, then they “live” in the same plane; and \( \vec{v} \times \vec{w} \) is orthogonal to that plane, so the determinant is zero (as desired).

When \( \vec{u}, \vec{v}, \vec{w} \) are linearly independent, then \( \vec{v} \times \vec{w} \) is NOT orthogonal to \( \vec{u} \), and the determinant is non-zero.
Formula Overload!

So ponder:

\[
\det(A) = \vec{u} \cdot (\vec{v} \times \vec{w})
\]

\[
= \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \cdot \left( \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix} \times \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} \right)
\]

\[
= \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix} \cdot \begin{bmatrix} a_{22}a_{33} - a_{32}a_{23} \\ a_{32}a_{13} - a_{12}a_{33} \\ a_{12}a_{23} - a_{22}a_{13} \end{bmatrix}
\]

\[
= a_{11}(a_{22}a_{33} - a_{32}a_{23}) + a_{21}(a_{32}a_{13} - a_{12}a_{33}) + a_{31}(a_{12}a_{23} - a_{22}a_{13})
\]

\[
= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}
\]

Let nightmares ensue!
Theorem (Sarrus’ Rule)

To find the determinant of a $3 \times 3$ matrix $A$, write the first 2 columns of $A$ to the right of $A$, then multiply entries along the six diagonals shown:

Add the (blue / right-down) and subtract the (red / left-down) products, to get

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

Unfortunately, Sarrus’ Rule does not generalize* to $n \times n$ matrices ($n > 3$).

* This tends to lead to lost points on midterms and finals...
The Determinant of an $n \times n$ Matrix

For an $n \times n$ matrix $A$:

- Let a **pattern**, $P$, of $A$ be a subset (vector) containing $n$ entries $a_{ij}$ selected from the matrix $A$ so that we have exactly one entry from each row and column. (There are $n(n-1)(n-2)\cdots1 = n!$ “$n$-factorial” such patterns.)

- Let $\text{prod}(P)$ be the product of all entries in a pattern $P$.

- An **inversion** in a pattern occurs when an entry is to the right, and above another; e.g. if the pattern contains $a_{12}$, and $a_{31}$ we have an inversion.

- Let $\text{sgn}(P)$ be the signature of the pattern, defined as $(-1)^{\#\text{inversions of } P}$.

What the @@#%&***??!! Let’s go back and apply this to the $3 \times 3$ case.
### Note: Fun with Factorials

They Grow **FAST!**

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Bottom line: there are *lots* of patterns for (not very) large matrices.

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Lecture Notes #6.1 — Determinants — (14/25)
Patterns, Inversions, Products, and Signatures in the $3 \times 3$ case...

We have established that

$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

No inversions.

2 Inversions:

$a_{31}:a_{12}, a_{31}:a_{23}$

3 Inversions:

$a_{31}:a_{22}, a_{31}:a_{13}, a_{22}:a_{31}$

1 Inversion:

$a_{32}:a_{23}$

1 Inversion:

$a_{21}:a_{12}$

2 Inversions:

$a_{21}:a_{13}, a_{32}:a_{13}$
The Determinant of an $n \times n$ Matrix

We can now define the determinant of an $n \times n$ matrix using its associated patterns (and the products, inversions and signatures of those patterns):

**Definition (The Determinant of an $n \times n$ Matrix)**

$$
\det(A) = \sum_{\text{All } n! \text{ patterns of } A} \text{sgn}(P) \text{prod}(P)
$$
All of what we have stated is true, but a big chunk of it is not really computationally practical for general matrices.

In the next lecture we will look at more properties of the determinant, and introduce structured (practical) ways to compute the determinant.

But, first let’s collect some useful examples...
Example (Invertibility)

For which values of \( \lambda \) is the matrix

\[
A = \begin{bmatrix}
-\lambda & 1 & 1 \\
1 & -\lambda & -1 \\
1 & 1 & -\lambda
\end{bmatrix}
\]

invertible?

**Solution:** We compute

\[
\det(A) = -\lambda^3 + \lambda = \lambda(\lambda + 1)(\lambda - 1),
\]

which shows that \( A \) is invertible except when \( \lambda \in \{-1, 0, 1\} \).
Example (Swapping Columns)

Consider

$$ \det(A) = \det \left( \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix} \right) = \vec{u} \cdot (\vec{v} \times \vec{w}) $$

and

$$ \det(B) = \det \left( \begin{bmatrix} \vec{u} & \vec{w} & \vec{v} \end{bmatrix} \right) = \vec{u} \cdot (\vec{w} \times \vec{v}) $$

Since the cross-product is anti-commutative, we must have

$$ \det(A) = -\det(B). $$

This is true in general, swapping rows or columns will flip the sign of the determinant.
Triangular Matrices

Example (Upper/Lower Triangular Matrix)

Let

\[ A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \]

Then \( \det(A) = a_{11}a_{22}a_{33} \), and \( \det(B) = b_{11}b_{22}b_{33} \)

Theorem (Determinant of a Triangular Matrix)

The determinant of a triangular matrix is the product of the diagonal entries of the matrix.

In particular, the determinant of a diagonal matrix is the product of its diagonal entries.
Block Matrices

Example (The Determinant of a Block Matrix)

Find

\[ \det(M) = \det \begin{pmatrix}
  a_{11} & a_{12} & b_{11} & b_{12} \\
  a_{21} & a_{22} & b_{21} & b_{22} \\
  0 & 0 & c_{11} & c_{12} \\
  0 & 0 & c_{21} & c_{22}
\end{pmatrix} = \det \begin{pmatrix}
  A & B \\
  0 & C
\end{pmatrix} \]

There are only 4 patterns with non-zero products:

\[
\begin{pmatrix}
  a_{11} & a_{12} & b_{11} & b_{12} \\
  a_{21} & a_{22} & b_{21} & b_{22} \\
  0 & 0 & c_{11} & c_{12} \\
  0 & 0 & c_{21} & c_{22}
\end{pmatrix},
\begin{pmatrix}
  a_{11} & a_{12} & b_{11} & b_{12} \\
  a_{21} & a_{22} & b_{21} & b_{22} \\
  0 & 0 & c_{11} & c_{12} \\
  0 & 0 & c_{21} & c_{22}
\end{pmatrix},
\begin{pmatrix}
  a_{11} & a_{12} & b_{11} & b_{12} \\
  a_{21} & a_{22} & b_{21} & b_{22} \\
  0 & 0 & c_{11} & c_{12} \\
  0 & 0 & c_{21} & c_{22}
\end{pmatrix},
\begin{pmatrix}
  a_{11} & a_{12} & b_{11} & b_{12} \\
  a_{21} & a_{22} & b_{21} & b_{22} \\
  0 & 0 & c_{11} & c_{12} \\
  0 & 0 & c_{21} & c_{22}
\end{pmatrix}
\]
Now,

\[ \det(M) = a_{11}a_{22}c_{11}c_{22} - a_{11}a_{22}c_{12}c_{21} - a_{12}a_{21}c_{11}c_{22} + a_{12}a_{21}c_{12}c_{21} \]

\[ = (a_{11}a_{22} - a_{12}a_{21})(c_{11}c_{22} - c_{12}c_{21}) \]

\[ = \det(A)\det(C). \]
Theorem (Determinant of a Block Matrix)

If \( A \) and \( C \) are square matrices (not necessarily of the same size), and \( B \) and \( 0 \) are matrices of appropriate size, then

\[
\begin{align*}
\det \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} &= \det(A) \det(C), \\
\det \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} &= \det(A) \det(C).
\end{align*}
\]

WARNING

The formula

\[
\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A) \det(D) - \det(B) \det(C)
\]

does NOT always hold.

See https://en.wikipedia.org/wiki/Determinant#Block_matrices
Available on Learning Glass videos:
6.1 — 1, 5, 11, 13, 23, 25, 27, 31, 43, 45
### Lecture – Book Roadmap

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* Strang does not talk about the combinatorial (pattern) definition of the determinant.