1 Student Learning Objectives
   - SLOs: Determinants

2 Introduction
   - Detour: The Cross Product in $\mathbb{R}^3$
   - $3 \times 3$ Determinant
   - $n \times n$ Determinant

3 Moving Along...
   - Useful Examples and Results

4 Suggested Problems
   - Suggested Problems 6.1
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5 Supplemental Material
   - Metacognitive Reflection
   - Problem Statements 6.1
   - Additional Example: Determinant of Sparse $4 \times 4$ Matrix

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Lecture Notes #6.1 — Determinants
After this lecture you should:

- Know that a Square Matrix has a Non-Zero Determinant if and only if it is Invertible

- Be familiar with the Connection between the determinant of a $3 \times 3$ matrix and the Cross Product (especially for Engineering / Physics students)

- Be familiar with Definition of the Determinant using the Combinatorial “Pattern” approach, and be able to use this definition to compute determinants of sparse matrices (i.e. matrices that have LOTS of zero-entries)
The matrix

\[ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \]

is invertible if and only if [Notes 2.4]

\[ \det(A) = ad - bc \neq 0. \]

The quantity \( ad - bc \) is called the determinant of the matrix \( A \).

It is natural to ask: *Can we assign a number \( \det(A) \) to any square matrix \( A \), such that \( A \) is invertible if-and-only-if \( \det(A) \neq 0 \)?*

To our euphoric joy, the answer is “yes!”
The Determinant of a $3 \times 3$ Matrix

First, we “upsize” to the $3 \times 3$ case:

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = [\vec{u} \ \vec{v} \ \vec{w}]$$

The matrix is not invertible if the three column vectors are contained in a same plane $\iff$ they are linearly dependent.

In the $3 \times 3$ case it is popular to express the determinant in terms of the *cross product*...

In the $n \times n$ ($n \geq 4$) setting, we tend to use the determinant to define a generalized cross product. The generalization is only called a *cross product* when $n = 7$. 

Peter Blomgren, {blomgren.peter@gmail.com} Lecture Notes #6.1 — Determinants — (5/45)
The 7D cross product is a \textit{bilinear} operation on vectors in seven-dimensional Euclidean space. For any $\vec{a}, \vec{b} \in \mathbb{R}^7$ it assigns a vector $\vec{v} = \vec{a} \times \vec{b}$, $\vec{v} \in \mathbb{R}^7$.

Like the 3D cross product, the 7D cross product is \textit{anticommutative} and $\vec{a} \times \vec{b} \perp \text{span}(\vec{a}, \vec{b})$.

Unlike in three dimensions, it does not satisfy the \textit{Jacobi identity}, and while the 3D cross product is unique up to a sign, there are many seven-dimensional cross products.

The 7D cross product has the same relationship to the \textit{octonions} as the three-dimensional product does to the \textit{quaternions} (a number system that extends to complex numbers).
The 7D cross product is one way of generalizing the cross product to other than 3D, and it is the only other non-trivial bilinear product of two vectors that is

(i) vector-valued,

(ii) anticommutative, and

(iii) orthogonal.

In other dimensions there are vector-valued products of three or more vectors that satisfy these conditions, and binary products with bivector results.
This is not meant to be useful... but it may be

**Binary Operation:** *(Mathematics)* a function of two variables. *Examples* +, −, ∗, /...

**Bilinear Operation:** a function which combines two arguments, and is linear in each of its arguments (when the other argument is kept fixed). *Old examples:* Matrix multiplication, Dot product.

**Anticommutative Operation:** a function of two arguments; which changes sign of the order of the arguments is reversed. *f(x, y) = −f(y, x).* *Old example:* subtraction *(a − b) = −(b − a).*
Jacobi Identity: a property of a binary operation which describes how the order of evaluation (the placement of parentheses in a multiple product) affects the result of the operation:

A binary operation $\circ$ on a set $S$ possessing a binary operation $+$ with an additive identity denoted by $0$ satisfies the Jacobi identity if:

$$a \circ (b \circ c) + b \circ (c \circ a) + c \circ (a \circ b) = 0 \quad \forall \ a, b, c \in S.$$ 

By contrast, for operations with the associative property, any order of evaluation gives the same result (parentheses in a multiple product are not needed).

Named after the German mathematician Carl Gustav Jakob Jacobi.
This is not meant to be useful... but it may be

**Real Numbers:** basis — \( \{1\} \), with multiplication table:

\[
\begin{array}{c|c}
\times & 1 \\
\hline
1 & 1 \\
\end{array}
\]

**Complex Numbers:** basis — \( \{1, i\} \), with multiplication table:

\[
\begin{array}{c|cc}
\times & 1 & i \\
\hline
1 & 1 & i \\
i & i & -1 \\
\end{array}
\]

**Quaternions:** basis — \( \{1, i, j, k\} \), with multiplication table:

\[
\begin{array}{c|ccccc}
\times & 1 & i & j & k \\
\hline
1 & 1 & i & j & k \\
i & i & -1 & k & -j \\
j & j & -k & -1 & i \\
k & k & j & -i & -1 \\
\end{array}
\]
The 7-Dimensional Cross Product

Crazy-Math-Interlude

[Focus :: Math]

This is not meant to be useful... it's just for “fun!”

Octonions: basis — \{1, i, j, k, ℓ, m, n, o\}, with multiplication table:

\[
\begin{array}{cccccccc}
\times & 1 & i & j & k & ℓ & m & n & o \\
1 & 1 & i & j & k & ℓ & m & n & o \\
i & i & -1 & k & -j & m & -ℓ & -o & n \\
j & j & -k & -1 & i & n & o & -ℓ & -m \\
k & k & j & -i & -1 & o & -n & m & -ℓ \\
ℓ & ℓ & -m & -n & -o & -1 & i & j & k \\
m & m & ℓ & -o & n & -i & -1 & -k & j \\
n & n & o & ℓ & -m & -j & k & -1 & -i \\
o & o & -n & m & ℓ & -k & -j & i & -1 \\
\end{array}
\]
3D Cross Product: basis — \{i, j, k\} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3\}, with multiplication table:

\[
\begin{array}{ccc}
\times & \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\
\vec{e}_1 & 0 & \vec{e}_3 & -\vec{e}_2 \\
\vec{e}_2 & -\vec{e}_3 & 0 & \vec{e}_1 \\
\vec{e}_3 & \vec{e}_2 & -\vec{e}_1 & 0 \\
\end{array}
\]
The 7-Dimensional Cross Product

Crazy-Math-Interlude

This is not meant to be useful... it’s just for “fun!”

7D Cross Product: *basis* —

\[ \{i, j, k, \ell, m, n, o\} = \{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4, \vec{e}_5, \vec{e}_6, \vec{e}_7\}, \text{ with multiplication table:} \]

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**Definition (Cross Product in \( \mathbb{R}^3 \))**

The cross product \( \vec{a} \times \vec{b} \) for \( \vec{a}, \vec{b} \in \mathbb{R}^3 \) is the vector in \( \mathbb{R}^3 \) with the following properties:

- \( \vec{a} \times \vec{b} \) is orthogonal to both \( \vec{a} \) and \( \vec{b} \).
- \( \| \vec{a} \times \vec{b} \| = \sin(\theta) \| \vec{a} \| \| \vec{b} \| ; \theta \) is the angle between \( \vec{a} \) and \( \vec{b} \); \( \theta \in [0, \pi] \).*
- The direction of \( \vec{a} \times \vec{b} \) follows the **right-hand-rule**

**Figure:** The right-hand-rule.

**Figure:** *This means that \( \| \vec{a} \times \vec{b} \| \) is the area of the parallelogram spanned by \( \vec{a} \) and \( \vec{b} \).
Theorem (Properties of the Cross Product)

The following equations hold \( \forall \vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^3, \) and \( \forall k \in \mathbb{R} \):

a. \( \vec{v} \times \vec{w} = -\vec{w} \times \vec{v}. \) \( \text{(anti-commutative)} \)

b. \( (k\vec{v}) \times \vec{w} = k(\vec{v} \times \vec{w}) = \vec{v} \times (k\vec{w}). \)

c. \( \vec{v} \times (\vec{u} + \vec{w}) = \vec{v} \times \vec{u} + \vec{v} \times \vec{w} \)

d. \( \vec{v} \times \vec{w} = \vec{0} \text{ if and only if } \vec{v} \parallel \vec{w}. \)

e. \( \vec{v} \times \vec{v} = \vec{0}. \)

f. \( \vec{e}_1 \times \vec{e}_2 = \vec{e}_3, \vec{e}_2 \times \vec{e}_3 = \vec{e}_1, \vec{e}_3 \times \vec{e}_1 = \vec{e}_2. \) \( \text{(right-hand-rule)} \)
Slight Detour: The Cross Product in $\mathbb{R}^3$

- The Dot (“inner”) Product, $\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$
  - $\vec{a} \cdot \vec{b} = \vec{a}^T \vec{b}$ is a scalar
  - $\vec{a} \cdot \vec{b} = \cos \theta \|\vec{a}\| \|\vec{b}\|$

- The Cross Product, $\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$
  - $\vec{a} \times \vec{b}$ is a vector $\perp \text{span}(\vec{a}, \vec{b})$
  - $\|\vec{a} \times \vec{b}\| = \sin(\theta) \|\vec{a}\| \|\vec{b}\|$

- The Outer Product, $\mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$
  - $\vec{a} \vec{b}^T = M$ is a matrix, $M \in \mathbb{R}^{n \times n}$
  - $m_{ij} = a_i b_j$

$\theta$ is the angle between $\vec{a}$ and $\vec{b}$; $\theta \in [0, \pi]$.

∃ Movie: Cross_Dot.mpeg
Theorem (The Cross Product in Components)

\[
\begin{bmatrix}
v_1 \\
v_2 \\
v_3 \\
\end{bmatrix} \times \begin{bmatrix}
w_1 \\
w_2 \\
w_3 \\
\end{bmatrix} = \begin{bmatrix}
v_2 w_3 - v_3 w_2 \\
v_3 w_1 - v_1 w_3 \\
v_1 w_2 - v_2 w_1 \\
\end{bmatrix}
\]

OK, back to determinants...
The $3 \times 3$ Determinant via the Cross Product

In the context of $3 \times 3$ matrices, we can compute

$$\det \left( \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix} \right) = \vec{u} \cdot (\vec{v} \times \vec{w})$$

When $\vec{u}, \vec{v}, \vec{w}$ are linearly dependent, then they “live” in the same plane; and $\vec{v} \times \vec{w}$ is orthogonal to that plane, so the determinant is zero (as desired).

When $\vec{u}, \vec{v}, \vec{w}$ are linearly independent, then $\vec{v} \times \vec{w}$ is NOT orthogonal to $\vec{u}$, and the determinant is non-zero.
Formula Overload!

So ponder:

\[
\begin{align*}
\text{det}(A) & = \vec{u} \cdot (\vec{v} \times \vec{w}) \\
& = \begin{vmatrix}
a_{11} \\
a_{21} \\
a_{31}
\end{vmatrix} \cdot \left( \begin{vmatrix}
a_{12} \\
a_{22} \\
a_{32}
\end{vmatrix} \times \begin{vmatrix}
a_{13} \\
a_{23} \\
a_{33}
\end{vmatrix} \right) \\
& = \begin{vmatrix}
a_{11} \\
a_{21} \\
a_{31}
\end{vmatrix} \cdot \begin{vmatrix}
a_{22}a_{33} - a_{32}a_{23} \\
a_{32}a_{13} - a_{12}a_{33} \\
a_{12}a_{23} - a_{22}a_{13}
\end{vmatrix} \\
& = a_{11}(a_{22}a_{33} - a_{32}a_{23}) + a_{21}(a_{32}a_{13} - a_{12}a_{33}) + a_{31}(a_{12}a_{23} - a_{22}a_{13}) \\
& = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\
& \quad - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}
\end{align*}
\]

Let nightmares ensue!
Theorem (Sarrus’ Rule)

To find the determinant of a $3 \times 3$ matrix $A$, write the first 2 columns of $A$ to the right of $A$, then multiply entries along the six diagonals shown:

\[
\begin{array}{ccc|ccc}
   a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\
   a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\
   a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \\
\end{array}
\]

Add the (blue / right-down) and subtract the (red / left-down) products, to get

\[
\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}
\]

Unfortunately, Sarrus’ Rule does not generalize* to $n \times n$ matrices ($n > 3$).

* This tends to lead to lost points on midterms and finals...
A “Fun” Definition of Determinants

The following discussion introduces the

- “Pattern,” or
- “Combinatorial” ← USEFUL FOR INTERNET SEARCHING…

definition of matrix determinants.
The Determinant of an $n \times n$ Matrix

For an $n \times n$ matrix $A$:

- Let a **pattern**, $P$, of $A$ be a subset (vector) containing $n$ entries $a_{ij}$ selected from the matrix $A$ so that we have exactly one entry from each row and column. (There are $n(n-1)(n-2)\cdots1 = n!$ “$n$-factorial” such patterns.)

- Let $\text{prod}(P)$ be the product of all entries in a pattern $P$.

- An **inversion** in a pattern occurs when an entry is to the right, and above another; e.g. if the pattern contains $a_{12}$, and $a_{31}$ we have an inversion.

- Let $\text{sgn}(P)$ be the signature of the pattern, defined as $(-1)^{\#\text{inversions of } P}$.

What the $@@@#$%&***????!!!$ Let’s go back and apply this to the $3 \times 3$ case.
Note: Fun with Factorials

They Grow **FAST!**

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</tr>
<tr>
<td>8</td>
<td>40,320</td>
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<td>9</td>
<td>362,880</td>
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<td>10</td>
<td>3,628,800</td>
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</tbody>
</table>

Bottom line: there are *lots* of patterns for (not very) large matrices.
Patterns, Inversions, Products, and Signatures in the 3 × 3 case...

We have established that

\[
\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}
\]

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

No inversions.

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

2 Inversions:
\[a_{31} : a_{12}, \ a_{31} : a_{23}\]

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

2 Inversions:
\[a_{21} : a_{13}, \ a_{32} : a_{13}\]

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

1 Inversion:
\[a_{32} : a_{23}\]

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

1 Inversion:
\[a_{21} : a_{12}\]

\[
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
\]

3 Inversions:
\[a_{31} : a_{22}, \ a_{31} : a_{13}, \ a_{22} : a_{31}\]
The Determinant of an $n \times n$ Matrix

We can now define the determinant of an $n \times n$ matrix using its associated patterns (and the products, inversions and signatures of those patterns):

Definition (The Determinant of an $n \times n$ Matrix)

$$\det(A) = \sum_{\text{All } n! \text{ patterns of } A} \text{sgn}(P) \text{prod}(P)$$
All of what we have stated is true, but a big chunk of it is not really computationally practical for general matrices.

However, for matrices with lots of zeros ("Sparse Matrices") this approach can be time-saving, since any pattern containing a zero will not contribute to the determinant.

Further, most properties (which we will discuss) of determinants can be traced back to the combinatorial definition; and some computational strategies leverage those properties.

In the next lecture we will look at more properties of the determinant, and introduce structured (practical) ways to compute the determinant.

But, first let’s collect some useful examples...
Example (Invertibility)

For which values of $\lambda$ is the matrix

$$A = \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & -1 \\ 1 & 1 & -\lambda \end{bmatrix}$$

invertible?

**Solution:** We compute

$$\det(A) = -\lambda^3 + \lambda = \lambda(\lambda + 1)(\lambda - 1),$$

which shows that $A$ is invertible except when $\lambda \in \{-1, 0, 1\}$.
Example (Swapping Columns)

Consider

\[
\det(A) = \det \left( \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix} \right) = \vec{u} \cdot (\vec{v} \times \vec{w})
\]

and

\[
\det(B) = \det \left( \begin{bmatrix} \vec{u} & \vec{w} & \vec{v} \end{bmatrix} \right) = \vec{u} \cdot (\vec{w} \times \vec{v})
\]

Since the cross-product is anti-commutative, we must have

\[
\det(A) = -\det(B).
\]

This is true in general, swapping rows or columns will flip the sign of the determinant.
Example (Upper/Lower Triangular Matrix)

Let

\[
A = \begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  0 & a_{22} & a_{23} \\
  0 & 0 & a_{33}
\end{bmatrix}, \quad B = \begin{bmatrix}
  b_{11} & 0 & 0 \\
  b_{21} & b_{22} & 0 \\
  b_{31} & b_{32} & b_{33}
\end{bmatrix}.
\]

Then \(\det(A) = a_{11}a_{22}a_{33}\), and \(\det(B) = b_{11}b_{22}b_{33}\).

Theorem (Determinant of a Triangular Matrix)

The determinant of a triangular matrix is the product of the diagonal entries of the matrix.

In particular, the determinant of a diagonal matrix is the product of its diagonal entries.
Example (The Determinant of a Block Matrix)

Find

$\det(M) = \det \begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ 0 & 0 & c_{11} & c_{12} \\ 0 & 0 & c_{21} & c_{22} \end{pmatrix} = \det \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$

There are only 4 patterns with non-zero products:

- $\begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ 0 & 0 & c_{11} & c_{12} \\ 0 & 0 & c_{21} & c_{22} \end{pmatrix}$
- $\begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ 0 & 0 & c_{11} & c_{12} \\ 0 & 0 & c_{21} & c_{22} \end{pmatrix}$
- $\begin{pmatrix} a_{11} & a_{12} & b_{11} & b_{12} \\ a_{21} & a_{22} & b_{21} & b_{22} \\ 0 & 0 & c_{11} & c_{12} \\ 0 & 0 & c_{21} & c_{22} \end{pmatrix}$
Now,

$$\det(M) = a_{11}a_{22}c_{11}c_{22} - a_{11}a_{22}c_{12}c_{21} - a_{12}a_{21}c_{11}c_{22} + a_{12}a_{21}c_{12}c_{21}$$

$$= (a_{11}a_{22} - a_{12}a_{21})(c_{11}c_{22} - c_{12}c_{21})$$

$$= \det(A) \det(C).$$
Block Matrices

**Theorem (Determinant of a Block Matrix)**

If $A$ and $C$ are square matrices (not necessarily of the same size), and $B$ and $0$ are matrices of appropriate size, then

$$\det \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \det(A) \det(C), \quad \det \begin{bmatrix} A & 0 \\ B & C \end{bmatrix} = \det(A) \det(C).$$

**WARNING**

The formula

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(A) \det(D) - \det(B) \det(C)$$

does NOT always hold.

See [https://en.wikipedia.org/wiki/Determinant#Block_matrices](https://en.wikipedia.org/wiki/Determinant#Block_matrices)
Available on Learning Glass videos:
6.1 — 1, 5, 11, 13, 23, 25, 27, 31, 43, 45
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* Strang does not talk about the combinatorial (pattern) definition of the determinant.
### Metacognitive Exercise — Thinking About Thinking & Learning

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(6.1.1) Is the matrix

\[ A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \]

invertible?

(6.1.5) Is the matrix

\[ A = \begin{bmatrix} 2 & 5 & 7 \\ 0 & 11 & 7 \\ 0 & 0 & 5 \end{bmatrix} \]

invertible?
(6.1.11), (6.1.13)

(6.1.11) For which values of \( k \) is the matrix

\[
A = \begin{bmatrix} k & 2 \\ 3 & 4 \end{bmatrix}
\]

invertible?

(6.1.13) For which values of \( k \) is the matrix

\[
A = \begin{bmatrix} k & 3 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 4 \end{bmatrix}
\]

invertible?
(6.1.23) Use the determinant to find out for which values of \( \lambda \) the matrix \((A - \lambda I_n)\)

\[
A = \begin{bmatrix}
1 & 2 \\
0 & 4 \\
\end{bmatrix}
\]

is invertible?

(6.1.25) Use the determinant to find out for which values of \( \lambda \) the matrix \((A - \lambda I_n)\)

\[
A = \begin{bmatrix}
4 & 2 \\
4 & 6 \\
\end{bmatrix}
\]

is invertible?
(6.1.27), (6.1.31)

(6.1.27) Use the determinant to find out for which values of \( \lambda \) the matrix \((A - \lambda I_n)\)

\[
A = \begin{bmatrix}
2 & 0 & 0 \\
5 & 3 & 0 \\
7 & 6 & 4 \\
\end{bmatrix}
\]

is invertible?

(6.1.31) Find the determinant of

\[
A = \begin{bmatrix}
1 & 9 & 8 & 7 \\
0 & 2 & 9 & 6 \\
0 & 0 & 3 & 5 \\
0 & 0 & 0 & 4 \\
\end{bmatrix}
\]
(6.1.43), (6.1.45)

(6.1.43) If $A$ is an $n \times n$ matrix, what is the relationship between
\[ \det(A) \quad \text{and} \quad \det(-A)? \]

(6.1.45) If $A$ is an $2 \times 2$ matrix, what is the relationship between
\[ \det(A) \quad \text{and} \quad \det(A^T)? \]
The combinatorial “pattern” approach can be computationally efficient when there are lots of zeros in the matrix; patterns with a 0 will not contribute to the value of the determinant, so we can skip exploring those patterns...

We consider computing the determinant of

\[
A = \begin{bmatrix}
0 & 0 & 1 & 2 \\
0 & 1 & 1 & 3 \\
1 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 \\
\end{bmatrix}
\]

Patterns including \(a_{31} = 1\):

\[
\begin{bmatrix}
x & 0 & 1 & 2 \\
x & 1 & 1 & 3 \\
\text{1} & x & x & x \\
x & 1 & 0 & 0 \\
\end{bmatrix}
\]
[continued] Patterns including \( a_{31} = 1 \):

\[
\begin{bmatrix}
  x & x & 1 & 2 \\
  x & 1 & x & x \\
  1 & x & x & x \\
  x & x & 0 & 0 \\
\end{bmatrix}
\]

\[P=\{a_{31}, a_{22}, \ldots\}\]

or

\[
\begin{bmatrix}
  x & x & 1 & 2 \\
  x & x & 1 & 3 \\
  1 & x & x & x \\
  x & 1 & x & x \\
\end{bmatrix}
\]

\[P=\{a_{31}, a_{42}, \ldots\}\]

We notice that for the left pattern, the next non-zero choice is \( a_{13} = 1 \), but that leaves \( a_{44} = 0 \) for the final selection; so this is a zero-end pattern. The right pattern can be competed in two ways:

\[
\begin{bmatrix}
  x & x & \textcircled{1} & x \\
  x & x & x & 3 \\
  1 & x & x & x \\
  x & 1 & x & x \\
\end{bmatrix}
\]

\[P=\{a_{31}, a_{42}, a_{13}, a_{24}\}\]

or

\[
\begin{bmatrix}
  x & x & x & \textcircled{2} \\
  x & x & \textcircled{1} & x \\
  1 & x & x & x \\
  x & 1 & x & x \\
\end{bmatrix}
\]

\[P=\{a_{31}, a_{42}, a_{23}, a_{14}\}\]
Patterns including $a_{31} = 1$:

The pattern $P = \{a_{31}, a_{42}, a_{13}, a_{24}\}$ has 4 inversions, so $\text{sgn}(P) = 1$, and $\text{prod}(P) = 1 \cdot 1 \cdot 1 \cdot 3 = 3$. Contribution to $\det(A) :: (+3)$.

The pattern $P = \{a_{31}, a_{42}, a_{23}, a_{14}\}$ has 5 inversions, so $\text{sgn}(P) = -1$, and $\text{prod}(P) = 1 \cdot 1 \cdot 1 \cdot 2 = 2$. Contribution to $\det(A) :: (-2)$. 
Next, we have to consider **Patterns including** $a_{41} = 2$:

\[
\begin{bmatrix}
  x & 0 & 1 & 2 \\
  x & 1 & 1 & 3 \\
  x & 0 & 0 & 0 \\
  2 & x & x & x \\
\end{bmatrix}
\]

There's only 1 non-zero choice in the 2nd column; and then in third: which forces us to pick a zero in the 4th column...

\[
\begin{bmatrix}
  x & x & 1 & 2 \\
  x & 1 & x & x \\
  x & x & 0 & 0 \\
  2 & x & x & x \\
\end{bmatrix} ,
\begin{bmatrix}
  x & x & 1 & x \\
  x & 1 & x & x \\
  x & x & x & 0 \\
  2 & x & x & x \\
\end{bmatrix} ,
\begin{bmatrix}
  x & x & 1 & x \\
  x & 1 & x & x \\
  x & x & x & 0 \\
  2 & x & x & x \\
\end{bmatrix}
\]

Since the pattern $P = \{a_{41}, a_{22}, a_{13}, a_{34}\} = \{2, 1, 1, 0\}$ contains a zero, we have $\text{prod}(P) = 2 \cdot 1 \cdot 1 \cdot 0 = 0$, it does not contribute to $\det(A)$.  

*Peter Blomgren, ⟨blomgren.peter@gmail.com⟩*
In summary, the determinant of

\[
A = \begin{bmatrix}
0 & 0 & 1 & 2 \\
0 & 1 & 1 & 3 \\
1 & 0 & 0 & 0 \\
2 & 1 & 0 & 0 \\
\end{bmatrix}
\]

is given by

\[
\det(A) = \text{sgn}(P_1) \prod(P_1) + \text{sgn}(P_2) \prod(P_2) = (+1)(+3)+(-1)(+2) = 1
\]

where

\[
P_1 = \{a_{31}, a_{42}, a_{13}, a_{24}\} = \{1, 1, 1, 3\},
\]

\[
P_2 = \{a_{31}, a_{42}, a_{23}, a_{14}\} = \{1, 1, 1, 2\}
\]

are the only 2 (out of 24) patterns with non-zero contributions.