Properties of the Determinant — (1/29)
1 Student Learning Objectives
   - SLOs: Properties of the Determinant

2 Properties of the Determinant
   - Transpose, and Linearity...
   - Computational Feasibility of the Determinant

3 More Properties
   - Invertibility, Gauss-Jordan Elimination, Products, Powers
   - Similar Matrices, Inverses

4 Minors, and Laplace (co-factor) Expansion
   - Minors
   - Laplace (co-factor) Expansion

5 Suggested Problems
   - Suggested Problems 6.2
   - Lecture–Book Roadmap
After this lecture you should:

- Know the Impact of Row Divisions/Swaps/Additions on the value of the Determinant
- Be familiar with computation of the Determinant of Products, Powers, Transposes, and Inverses of matrices: \( \det(AB), \det(A^k), \det(A^T), \) and \( \det(A^{-1}) \)
- Be able to compute the determinant using
  - Combinatorial “Pattern” approach, [NOTES 6.1]
  - Row Reductions,
  - Laplace (co-factor) Expansion Method [“TRADITIONAL” WAY].
Consider the patterns, $P$ which we use to define the determinant; e.g.

$$A = \begin{bmatrix} \cdot & \cdot & * & \cdot & \cdot \\ * & \cdot & \cdot & \cdot & \cdot \\ \cdot & * & \cdot & \cdot & \cdot \\ * & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & * & \cdot & \cdot \end{bmatrix}, \quad A^T = \begin{bmatrix} \cdot & * & \cdot & \cdot & \cdot \\ * & \cdot & \cdot & \cdot & \cdot \\ \cdot & * & \cdot & \cdot & \cdot \\ * & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & * & \cdot & \cdot \end{bmatrix}$$

- Clearly, the product of the pattern $\text{prod}(P)$ is preserved in the transposed pattern;
- The number of inversions is preserved: the down-left number and up-right number just switch roles — therefore $\text{sgn}(P^T) = \text{sgn}(P)$.
- Since this is true for all patterns we must have $\det(A^T) = \det(A)$. 

Peter Blomgren, \{blomgren.peter@gmail.com\}  Properties of the Determinant — (4/29)
Theorem (Determinant of the Transpose)

If \( A \) is a square matrix, then

\[
\det(A^T) = \det(A).
\]

This means that any property expressed in terms of columns/rows is also true for rows/columns;

- e.g. last time we saw that swapping two columns in a 3-by-3 matrix changed the sign of the determinant;

- so, by the above, it directly follows that swapping two rows in a 3-by-3 matrix also changes the sign.
Theorem (Linearity of the Determinant in the Columns)

Consider fixed column vectors

\[ \vec{v}_1, \ldots, \vec{v}_{i-1}, \vec{v}_{i+1}, \ldots, \vec{v}_n \in \mathbb{R}^n. \]

Then the function \( T : \mathbb{R}^n \to \mathbb{R} \) defined by

\[ T(\vec{x}) = \det \left( \begin{bmatrix} \vec{v}_1 & \ldots & \vec{v}_{i-1} & \vec{x} & \vec{v}_{i+1} & \ldots & \vec{v}_n \end{bmatrix} \right) \]

is a linear transformation.

We can convince ourselves that the theorem is indeed true...
Linearity of the Determinant in the Columns

"Proof"

- We note that $\prod(P)$ is linear in all the rows and columns, since it contains exactly one factor from each row/column.

- The determinant is a sum of all the $\prod(P_k)$ (with appropriate sign given by $\text{sgn}(P_k)$ — i.e. The determinant is a linear combination of pattern products.

We express $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$, and $T(k\vec{x}) = kT(\vec{x})$:

$$\det \left( \begin{bmatrix} \vec{v}_1 & \ldots & \vec{x} + \vec{y} & \ldots & \vec{v}_n \end{bmatrix} \right) =$$

$$\det \left( \begin{bmatrix} \vec{v}_1 & \ldots & \vec{x} & \ldots & \vec{v}_n \end{bmatrix} \right) +$$

$$\det \left( \begin{bmatrix} \vec{v}_1 & \ldots & \vec{y} & \ldots & \vec{v}_n \end{bmatrix} \right)$$

$$\det \left( \begin{bmatrix} \vec{v}_1 & \ldots & k\vec{x} & \ldots & \vec{v}_n \end{bmatrix} \right) = k\det \left( \begin{bmatrix} \vec{v}_1 & \ldots & \vec{x} & \ldots & \vec{v}_n \end{bmatrix} \right)$$
Say you are faced with computing the determinant of a $32 \times 32$ matrix (not very large by modern standards). Using the “pattern”-method, such a computation would require $31 \cdot 32! \approx 8.16 \cdot 10^{36}$ multiplications.

Now say you have access to an “exaflop computer” — which can perform $10^{18}$ operations / second; then your computation would only take about $10^{18}$ seconds... how long is that?

- 100 seconds/minute $\mapsto 10^{16}$ minutes
- 100 minutes/hour $\mapsto 10^{14}$ hours
- 100 hours/day $\mapsto 10^{12}$ days
- 1000 days/year $\mapsto 10^9$ years.

so... only about 7% of the age of the universe.

Perfect midterm question, yeah?!?
How Fast is $10^{18}$ operations/s?

As of November 2016, according to https://www.top500.org/, the fastest supercomputer in the world:

<table>
<thead>
<tr>
<th>Site</th>
<th>National Supercomputing Center in Wuxi China</th>
</tr>
</thead>
<tbody>
<tr>
<td>System</td>
<td>Sunway TaihuLight - Sunway MPP, Sunway</td>
</tr>
<tr>
<td></td>
<td>SW26010 260C 1.45GHz, Sunway NRCPC</td>
</tr>
<tr>
<td>Cores</td>
<td>10,649,600</td>
</tr>
<tr>
<td>Rmax</td>
<td>93,014.6 TFlop/s</td>
</tr>
<tr>
<td>Rpeak</td>
<td>125,435.9 TFlop/s ($0.125 \times 10^{18}$ Flop/s)</td>
</tr>
<tr>
<td>Power</td>
<td>15,371 kW</td>
</tr>
</tbody>
</table>
Clearly “we” have to figure something out...

- We know that the determinant of an (upper) triangular matrix is the product of the diagonal entries.
- We can take a general matrix and apply row-reductions to generate an upper triangular matrix.
- Can we “glue” these ideas together?

The short answer is “yes.” The slightly longer answer requires us to figure out how the row-reduction operation changes (or don’t change) the value of the determinant.
Our three fundamental row-operations are:

1. **Row division**: Dividing a row by a non-zero scalar $k$.
2. **Row swap**: Swapping two rows.
3. **Row addition**: Adding (subtracting) a multiple of one row to another.
First, consider the 2 \times 2 case $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, with $\det(A) = ad - bc$:

1. **Row division:** If $B = \begin{bmatrix} a/k & b/k \\ c & d \end{bmatrix}$, then $\det(B) = ad/k - bc/k = \det(A)/k$.

   \[ \Rightarrow \text{Scaling of the Determinant} \]

2. **Row swap:** If $B = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$, then $\det(B) = cb - da = -\det(A)$.

   \[ \Rightarrow \text{Sign Change of the Determinant} \]

3. **Row addition:** If $B = \begin{bmatrix} a + kc & b + kd \\ c & d \end{bmatrix}$, then $\det(B) = (a + kc)d - (b + kd)c = (ad - bc) + k(cd - dc) = \det(A)$.

   \[ \Rightarrow \text{No Change to the Determinant} \]
Row-Reductions and Determinants: The \( n \times n \) Case

Next, we think about extending to the \( n \times n \) case:

1. **Row division**: This follows from the linearity!

2.a **Row swap**: This is a bit trickier: consider swapping two adjacent rows, and think about how this impacts ALL the patterns:
   - Each pattern has exactly one member in each row.
   - The left-most entry is either in the top row (no inversion in this piece of the pattern); or in the bottom row (one inversion).
   - After swapping (no inversion) → (one inversion)
   - (one inversion) → (no inversion)

Bottom line: adjacent swaps lead to a sign-flip of the determinant.
2.b Row swap: OK, what about non-adjacent swaps?

- For two rows \( n \) steps apart:
  - The Bottom row can “climb” to the top using \( n \) adjacent row-swaps;
  - then, the former Top row (now in the second row), can “sink” to the bottom location using \((n - 1)\) additional adjacent row-swaps:

\[
\begin{bmatrix}
\vec{r}_1^T \\
\vec{r}_2^T \\
\vdots \\
\vec{r}_n^T \\
\vec{r}_{n+1}^T
\end{bmatrix} \xrightarrow{n} \begin{bmatrix}
\vec{r}_{n+1}^T \\
\vec{r}_1^T \\
\vec{r}_2^T \\
\vdots \\
\vec{r}_n^T
\end{bmatrix} \xrightarrow{1} \begin{bmatrix}
\vec{r}_{n+1}^T \\
\vec{r}_2^T \\
\vec{r}_1^T \\
\vdots \\
\vec{r}_n^T
\end{bmatrix}
\]

- We get \((2n - 1)\) – an odd number – of row-swaps; i.e. and odd number of sign-flips; so the value of the determinant flips.

- Bottom line: Any row-swap leads to a sign-flip for the determinant.

- This means that if you have two equal rows, and swap them \( \det(A) = -\det(A) \), which means \( \det(A) = 0 \).
Row-Reductions and Determinants

- **Row addition:**

\[
A = \begin{bmatrix}
\ddots \\
\vec{v}_i^T \\
\vec{v}_j^T \\
\vdots
\end{bmatrix} \quad \rightarrow B = \begin{bmatrix}
\ddots \\
\vec{v}_i^T \\
\vec{v}_j^T + k\vec{v}_i^T \\
\vdots
\end{bmatrix}
\]

Now, we use linearity:

\[
\det\left(\begin{bmatrix}
\ddots \\
\vec{v}_i^T \\
\vec{v}_j^T + k\vec{v}_i^T \\
\vdots
\end{bmatrix}\right) = \det\left(\begin{bmatrix}
\ddots \\
\vec{v}_i^T \\
\vec{v}_j^T \\
\vdots
\end{bmatrix}\right) + k\det\left(\begin{bmatrix}
\ddots \\
\vec{v}_i^T \\
\vec{v}_j^T \\
\vdots
\end{bmatrix}\right)
\]

\[
= \det(A) + k \cdot 0
\]
Row-Reductions and Determinants

We summarize:

**Theorem (Elementary Row Operations and Determinants)**

- **a.** If $B$ is obtained from $A$ by dividing a row of $A$ by a scalar $k$, then
  \[ \det(B) = \frac{1}{k} \det(A) \]

- **b.** If $B$ is obtained from $A$ by a row swap, then
  \[ \det(B) = -\det(A) \]

- **c.** If $B$ is obtained from $A$ by adding a multiple of a row of $A$ to another row, then
  \[ \det(B) = \det(A) \]

Analogous results hold for elementary column operations.
**Row-Reductions and Determinants**

**det(rref(A)) and det(A)**

Now, if we *in the process* of computing the reduced-row-echelon-form of a matrix $A$

- count the number of row-swaps: $s$, and
- keep track of scalar divisions $k_1, \ldots, k_r$.

then:

$$\det(\text{rref}(A)) = (-1)^s \frac{1}{k_1 k_2 \ldots k_r} \det(A),$$

or

$$\det(A) = (-1)^s k_1 k_2 \ldots k_r \det(\text{rref}(A))$$

This should save us a second or two... Computing $\text{rref}(A)$ of a $32 \times 32$ matrix will require no more than $\approx \frac{32^3}{3} \approx 11,000$ operations (which is slightly smaller than $8.16 \cdot 10^{36}$)
Figure: For small problems ($n \leq 4$), it is faster to use the “pattern” method (usually via the Laplace co-factor expansion), but for large ($n \geq 6$) problems using row-reductions is faster.
If $A$ is invertible, then $\text{rref}(A) = I_n$, so that $\det(\text{rref}(A)) = \det(I_n) = 1$, and

$$\det(A) = (-1)^s k_1 k_2 \ldots k_r \neq 0.$$  

If $A$ is non-invertible, then the last row of $\text{rref}(A)$ is all zeros, and by linearity $\det(\text{rref}(A)) = 0$; so that $\det(A) = 0$.

**Theorem (Invertibility and Determinant)**

A square matrix $A$ is invertible if and only if $\det(A) \neq 0$.  

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If instead of computing $\det(\text{rref}(A))$, we perform elementary row operations on $A$ to transform it into some matrix $B$, and $\det(B)$ is easy to compute; the same rules apply; if we performed $s$ row swaps, and scaled rows by the factors $k_1, \ldots, k_r$, then

$$\det(A) = (-1)^s k_1 k_2 \ldots k_r \det(B)$$

Transforming $A$ into upper triangular form $U$ is a popular choice, since

$$\det(U) = \prod_{k=1}^{n} u_{kk}.$$
Theorem (Determinants of Products and Powers)

If $A$ and $B$ are $n \times n$ matrices, and $m \in \mathbb{Z}^+$ is a positive integer, then

a. $\det(AB) = (\det(A)) (\det(B))$, and

b. $\det(A^m) = (\det(A))^m$.

(a.)

(i — Assume $A$ is invertible): It is fairly straight-forward to convince yourself that the row-operations required to transform $A$ to $I_n$ applied to the augmented system $[ A \mid AB ]$ gives:

$$\text{rref} \left( \begin{bmatrix} A & AB \end{bmatrix} \right) = \begin{bmatrix} I_n & I_n B \end{bmatrix} = \begin{bmatrix} I_n & B \end{bmatrix}$$

continued...
Determinants of Products and Powers

(a.)

In the process we perform $s$ row-swaps, and divide various rows by $k_1, \ldots, k_r$, so

$$\det(A) = (-1)^s k_1 k_2 \ldots k_r$$

and

$$\det(AB) = (-1)^s k_1 k_2 \ldots k_r \det(B) = (\det(A)) (\det(B)).$$

(ii — When $A$ is not invertible): If $A$ is not invertible, then neither is $AB$, so

$$(\det(A)) (\det(B)) = 0 \cdot \det(B) = \det(AB).$$
(b.)

Apply part (a.) \((m - 1)\) times to get:

\[
\det(A^m) = (\det(A))^m
\]

Example (Similar Matrices)

Consider two similar matrices \(A, B\); where \(S\) is an invertible matrix so that

\[
AS = SB.
\]

The previous theorems then says

\[
\det(A) \det(S) = \det(S) \det(B),
\]

so that \(\det(A) = \det(B)\).
Determinants of Similar Matrices

Theorem (Determinants of Similar Matrices)

If a matrix $A$ is similar to $B$, then $\det(A) = \det(B)$.

Theorem (Determinant of an Inverse)

If a matrix $A$ is invertible, then

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Proof.

Since $I_n = A^{-1}A$, $1 = \det(I_n) = \det(A^{-1})\det(A)$. 

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Ponder, once again, the $3 \times 3$ case, and recall the result of Sarrus’ formula

\[ \det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \]

We can save 3 (of the 12) multiplications by writing

\[ \det(A) = a_{11}(a_{22}a_{33} - a_{23}a_{32}) + a_{21}(a_{13}a_{32} - a_{12}a_{33}) + a_{31}(a_{12}a_{23} - a_{13}a_{22}) \]

We can recognize this as

\[
\begin{align*}
\det(A) &= a_{11} \det \left( \begin{bmatrix}
\ast & a_{22} & a_{23} \\
\ast & a_{32} & a_{33}
\end{bmatrix} \right) - a_{21} \det \left( \begin{bmatrix}
\ast & a_{12} & a_{13} \\
\ast & a_{32} & a_{33}
\end{bmatrix} \right) + a_{31} \det \left( \begin{bmatrix}
\ast & a_{12} & a_{13} \\
\ast & a_{22} & a_{23}
\end{bmatrix} \right) \\
&= a_{11} \det \left( \begin{bmatrix}
a_{22} & a_{23} \\
\ast & a_{33}
\end{bmatrix} \right) - a_{21} \det \left( \begin{bmatrix}
a_{12} & a_{13} \\
\ast & a_{33}
\end{bmatrix} \right) + a_{31} \det \left( \begin{bmatrix}
a_{12} & a_{13} \\
\ast & a_{23}
\end{bmatrix} \right)
\end{align*}
\]
Definition (Minors)

For an \( n \times n \) matrix \( A \), let \( A_{ij} \) be the matrix obtained by omitting the \( i^{th} \) row, and \( j^{th} \) column of \( A \). The determinant of the \((n - 1) \times (n - 1)\) matrix \( A_{ij} \) is called a minor of \( A \).

With this language, the determinant of the \( 3 \times 3 \) matrix \( A \):

\[
\det(A) = a_{11} \det(A_{11}) - a_{21} \det(A_{21}) + a_{31} \det(A_{31}).
\]

This is know as the Laplace expansion, or co-factor expansion of \( \det(A) \) down the first column.

We generalize this common strategy...
Theorem (Laplace (co-factor) Expansion)

We can compute the determinant of an $n \times n$ matrix $A$ by Laplace expansion down any column, or along any row:

- **Expansion down the $j^{th}$ column:**

  \[
  \det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}).
  \]

- **Expansion along the $i^{th}$ row:**

  \[
  \det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}).
  \]
Available on Learning Glass videos:
6.2 — 1, 5, 7, 9, 11, 12, 13, 15
<table>
<thead>
<tr>
<th>Lecture</th>
<th>Book, [GS5–]</th>
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<tr>
<td>6.1*</td>
<td>§5.1, §5.2, §5.3</td>
</tr>
<tr>
<td>6.2</td>
<td>§5.1, §5.2, §5.3</td>
</tr>
<tr>
<td>6.3</td>
<td>§5.1, §5.2, §5.3</td>
</tr>
</tbody>
</table>

* Strang does not talk about the combinatorial (pattern) definition of the determinant.