1. **Student Learning Objectives**
   - SLOs: Determinant: Geometrical View, Cramer’s Rule

2. **The Determinant: Redux**
   - Geometrical Interpretations (“Volume”)
   - The Determinant as Expansion Factor
   - Cramer’s Rule: The Worst Idea In Linear Algebra

3. **Suggested Problems**
   - Suggested Problems 6.3
   - Lecture–Book Roadmap
After this lecture you should:

- Know that $|\det(A)| = 1 \iff$ Orthogonal Matrix; e.g. rotation or reflection.
- Be familiar with the Interpretation of the determinant as an $m$-Volume; and/or an expansion factor.
- Forget about Cramer’s Rule: Don’t Use It!
We have plenty of formulas describing the determinant; next we ponder its geometrical interpretations.

**Example (Determinant of an Orthogonal Matrix)**

What are the possible values of $\det(A)$, when $A$ is orthogonal?

**Answer:** Orthogonal means that $A^TA = I_n$, so we have

$$1 = \det(I_n) = \det(A^TA) = \det(A^T)\det(A) = (\det(A))^2,$$

therefore $\det(A) = \pm 1$. 
Orthogonal Matrices

Theorem (The Determinant of an Orthogonal Matrix)

The determinant of an orthogonal matrix is either 1 or \(-1\).

Example (The Determinant of a Rotation Matrix)

Let

\[
M(\theta) = \begin{bmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{bmatrix}
\]

Then

\[
\det(M(\theta)) = (\cos(\theta))^2 + (\sin(\theta))^2 = 1.
\]

Definition (Rotation Matrix)

An orthogonal \(n \times n\) matrix \(A\), with \(\det(A) = 1\) is called a \textit{rotation matrix}, and the linear transformation \(T(\vec{x}) = A\vec{x}\) is called a \textit{rotation}. 

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Example (The Determinant of a Reflection Matrix)

Let

\[ M(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix} \]

Then

\[ \det(M(\theta)) = -\cos^2(\theta) - \sin^2(\theta) = -1. \]

**Figure:** Reflections with \( M(4\pi/3) \), and \( M(7\pi/4) \).
We have given the $2 \times 2$ determinant a geometrical interpretation, with

$$\det(A) = \det \left( \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \right) = \sin(\theta) \| \vec{v}_1 \| \| \vec{v}_2 \|,$$

$|\det(A)|$ is the area of the parallelogram spanned by the vectors $\vec{v}_1$, and $\vec{v}_2$. $|\sin(\theta)| \| \vec{v}_2 \| = \| \vec{v}_2^\perp \vec{v}_1 \|$, and therefore

$$|\det(A)| = \| \vec{v}_1 \| \| \vec{v}_2^\perp \|.$$
We can use Gram-Schmidt ($QR$-factorization) to generalize to the $n \times n$ case:

- Given an invertible $n \times n$ matrix $A = [\vec{v}_1 \cdots \vec{v}_n]$, we can write $A = QR$ where $Q$ is an orthogonal matrix, and $R$ is an upper triangular matrix:

$$|\det(A)| = |\det(Q)| |\det(R)|$$

where $|\det(Q)| = 1$ since it is orthogonal.

- The diagonal entries of $R$ are given by

$$r_{11} = \|\vec{v}_1\|, \quad r_{jj} = \|\vec{v}_j^\perp\|, \quad j \geq 2$$

where $\|\vec{v}_j^\perp\|$ is the component of $\vec{v}_j$ perpendicular to $\text{span}(\vec{v}_1, \ldots, \vec{v}_{j-1})$. 
The Determinant as Area, Volume, and Generalized Volume

**Theorem (The Determinant in Terms of the Columns)**

If $A$ is an $n \times n$ with columns $\vec{v}_1, \ldots, \vec{v}_n$, then

$$|\text{det}(A)| = \|\vec{v}_1\| \|\vec{v}_2\| \cdots \|\vec{v}_n\|,$$

where $\|\vec{v}_j\|$ is the component of $\vec{v}_j$ perpendicular to $\text{span}(\vec{v}_1, \ldots, \vec{v}_{j-1})$.

*(Notation from the Gram-Schmidt process)*.

**Grammar-Schmidt QR-factorization** may seem like a painful thing (it requires about 2–3 times the work of just computing $\text{rref}(A)$, but it gives us a useful results:

1. $Q$ contains an orthonomal basis for $\text{im}(A)$
2. $R$ contains the transformation from (non-orthogonal) $A$-coordinates to (orthonormal) $Q$-coordinates.
3. $R$ can be used to compute $\text{det}(A)$ using only $(n-1)$ multiplications.
The Determinant as Area, Volume, and Generalized Volume

Figure: With \((\vec{v}_1, \vec{v}_2, \vec{v}_3) = (\vec{b}, \vec{c}, \vec{a})\), we have

\[
\text{Volume} = \|\vec{v}_1\| \|\vec{v}_2\| \|\vec{v}_3\| \cos \theta
\]

Base Area Height

Theorem (Volume of a Parallelepiped in \(\mathbb{R}^3\))

Consider a 3 \times 3 matrix \(A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]\). Then the volume of the parallelepiped defined by \(\vec{v}_1, \vec{v}_2, \text{ and } \vec{v}_3\) is \(|\det(A)|\).
Definition (Parallelepipeds in $\mathbb{R}^n$)

Consider the vectors $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$. The $m$-Parallelepiped defined by these vectors is:

$$\{ \vec{y} \in \mathbb{R}^n : \vec{y} = c_1 \vec{v}_1 + \cdots + c_m \vec{v}_m, \ c_k \in [0, 1]\}$$

The $m$-volume $V(\vec{v}_1, \ldots, \vec{v}_m)$ of this $m$-Parallelepiped is defined recursively by:

$$V(\vec{v}_1) = \|\vec{v}_1\|,$$

and

$$V(\vec{v}_1, \ldots, \vec{v}_m) = V(\vec{v}_1, \ldots, \vec{v}_{m-1}) \|\vec{v}_m\|,$$

or equivalently

$$V(\vec{v}_1, \ldots, \vec{v}_m) = \|\vec{v}_1\| \|\vec{v}_2\| \cdots \|\vec{v}_m\|.$$
We can leverage the $QR$-factorization:

Let $A \in \mathbb{R}^{n \times m}$, with columns $\vec{v}_1, \ldots, \vec{v}_m$. If the columns are linearly independent, consider the $QR$-factorization $A = QR$. Then

$$A^T A = (QR)^T (QR) = (R^T Q^T)(QR) = R^T (Q^T Q) R = R^T R,$$

so that

$$\det(A^T A) = \det(R^T R) = (\det(R))^2 = (r_{11} r_{22} \cdots r_{mm})^2$$

$$= (\| \vec{v}_1 \| \| \vec{v}_2 \| \cdots \| \vec{v}_m \|)^2 = (V(\vec{v}_1, \ldots, \vec{v}_m))^2.$$
Consider the vectors $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$. Then the $m$-Volume of the $m$-Parallelepiped defined by these vectors is:

$$\sqrt{\det(A^T A)},$$

where $A$ is the $n \times m$ matrix with columns $\vec{v}_1, \ldots, \vec{v}_m$.

In particular, when $m = n$, this volume is

$$|\det(A)|.$$
Consider a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$. We have discussed how such a transform impacts lengths, and angles. For a transform $\mathbb{R}^2 \to \mathbb{R}^2$ it also makes sense to think about the 2-volume (aka “The Area”); and for $\mathbb{R}^n \to \mathbb{R}^n$ we can discuss the $m$-Volume(s).
The Determinant as Expansion Factor

We start in the $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ case, and let the “input area” ($\Omega$) be the unit square (with area 1), described by the two vectors $\vec{e}_1$, and $\vec{e}_2$.

The “output area” ($T(\Omega)$), is then described by $A\vec{e}_1 = \vec{v}_1$, and $A\vec{e}_2 = \vec{v}_2$, i.e. the parallelepiped spanned by the columns of $A$; here the area is $|\det(A)|$.

The expansion factor is

$$\frac{\text{area of } T(\Omega)}{\text{area of } \Omega} = \frac{|\det(A)|}{1} = |\det(A)|.$$
The Determinant as Expansion Factor

If the input parallelepiped is described by two vectors $\vec{w}_1$, and $\vec{w}_2$, then the original area is $|\det(B)|$, where $B = \begin{bmatrix} \vec{w}_1 & \vec{w}_2 \end{bmatrix}$.

The “output area” ($T(\Omega)$), is then described by $A\vec{w}_1 = \vec{v}_1$, and $A\vec{w}_2 = \vec{v}_2$; so the area of $T(\Omega)$ is given by

$$|\det(\begin{bmatrix} A\vec{w}_1 & A\vec{w}_2 \end{bmatrix})| = |\det(AB)| = |\det(A)| |\det(B)|.$$

The expansion factor is

$$\frac{\text{area of } T(\Omega)}{\text{area of } \Omega} = \frac{|\det(A)| |\det(B)|}{|\det(B)|} = |\det(A)|.$$
The Determinant as Expansion Factor

Theorem (Expansion Factor)

Consider a linear transformation \( T : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by \( T(\vec{x}) = A\vec{x} \). Then \( |\det(A)| \) is the expansion factor

\[
\frac{\text{area of } T(\Omega)}{\text{area of } \Omega}
\]

of \( T \) on parallelograms \( \Omega \).

Likewise, for linear transformations \( T : \mathbb{R}^n \to \mathbb{R}^n \) defined by \( T(\vec{x}) = A\vec{x} \), \( |\det(A)| \) is the expansion factor of \( T \) on \( n \)-parallelepipeds:

\[
V(A\vec{v}_1, \ldots, A\vec{v}_n) = |\det(A)| \cdot V(\vec{v}_1, \ldots, \vec{v}_n),
\]

for all vectors \( \vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n \).
The Determinant as Expansion Factor

Stating the “Obvious”

Since $\det(AB) = \det(A) \det(B)$, which of course holds for $B = A^{-1}$, the expansion factors satisfy:

$$|\det(AB)| = |\det(A)| |\det(B)|,$$

and

$$|\det(A^{-1})| = \frac{1}{|\det(A)|}.$$

Stating the “Less Obvious”

It is possible to show that the expansion factor associated with a linear transformation $T(\vec{x}) = A\vec{x}$ (as we have defined it) holds for any region $Ω$ (not just parallelograms or $n$-parallelepipeds).
Theorem (Cramer’s Rule)

Consider the linear system

\[ A\vec{x} = \vec{b} \]

where \( A \) is an invertible \( n \times n \) system. The components \( x_i \) of the solution vector \( \vec{x} \) are

\[ x_i = \frac{\det(A_{\vec{b},i})}{\det(A)}, \]

where \( A_{\vec{b},i} \) is the matrix obtained by replacing the \( i^{th} \) column of \( A \) by \( \vec{b} \).

Peter’s Postulate

Solving linear systems using Cramer’s Rule is a BAD IDEA. — We need to compute \((n + 1)\) determinants of size \((n \times n)\).
Recall:

**Definition (Minors)**

For an \( n \times n \) matrix \( A \), let \( A_{ij} \) be the matrix obtained by omitting the \( i^{th} \) row, and \( j^{th} \) column of \( A \). The determinant of the \((n - 1) \times (n - 1)\) matrix \( A_{ij} \) is called a *minor* of \( A \).

Now:

**Definition (The Classical Adjoint)**

The classical adjoint \( M = \text{adj}(A) \) of an invertible \( n \times n \) matrix, is the matrix whose \( ij^{th} \) entry \( m_{ij} = (-1)^{i+j} \det(A_{ij}) \). Yes, we have to compute \( n^2 \) \((n - 1) \times (n - 1)\) determinants to build the adjoint! With “only” one more \( n \times n \) determinant, we can express the inverse:

\[
A^{-1} = \frac{1}{\det(A)} \text{adj}(A).
\]
Available on Learning Glass videos:
6.3 — 1, 3, 5, 7, 9, 11, 13, 19 - 20 - 21
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* Strang does not talk about the combinatorial (pattern) definition of the determinant.