Outline

1. Student Learning Objectives
   - SLOs: Determinant: Geometrical View, Cramer’s Rule

2. The Determinant: Redux
   - Geometrical Interpretations ("Volume")
   - The Determinant as Expansion Factor
   - Cramer’s Rule: The Worst Idea In Linear Algebra

3. Suggested Problems
   - Suggested Problems 6.3
   - Lecture–Book Roadmap
After this lecture you should:

- Know that $|\text{det}(A)| = 1 \iff$ Orthogonal Matrix; e.g. rotation or reflection.
- Be familiar with the Interpretation of the determinant as an $m$-Volume; and/or an expansion factor.
- Forget about Cramer’s Rule: Don’t Use It!
We have plenty of formulas describing the determinant; next we ponder its geometrical interpretations.

**Example (Determinant of an Orthogonal Matrix)**

What are the possible values of $\det(A)$, when $A$ is orthogonal?

**Answer:** Orthogonal means that $A^T A = I_n$, so we have

$$1 = \det(I_n) = \det(A^T A) = \det(A^T) \det(A) = (\det(A))^2,$$

therefore $\det(A) = \pm 1$. 
Orthogonal Matrices

Theorem (The Determinant of an Orthogonal Matrix)

*The determinant of an orthogonal matrix is either 1 or −1.*

Example (The Determinant of a Rotation Matrix)

Let

\[ M(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \]

Then

\[ \det(M(\theta)) = (\cos(\theta))^2 + (\sin(\theta))^2 = 1. \]

Definition (Rotation Matrix)

An orthogonal \( n \times n \) matrix \( A \), with \( \det(A) = 1 \) is called a *rotation matrix*, and the linear transformation \( T(\vec{x}) = A\vec{x} \) is called a *rotation*.
Orthogonal Matrices

Example (The Determinant of a Reflection Matrix)

Let

$$M(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix}$$

Then

$$\det(M(\theta)) = -\cos^2(\theta) - \sin^2(\theta) = -1.$$
We have given the $2 \times 2$ determinant a geometrical interpretation, with

$$\det(A) = \det\left(\begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}\right) = \sin(\theta) \|\vec{v}_1\| \|\vec{v}_2\|,$$

$|\det(A)|$ is the area of the parallelogram spanned by the vectors $\vec{v}_1$, and $\vec{v}_2$. $|\sin(\theta)| \|\vec{v}_2\| = \|\vec{v}_2 \perp \vec{v}_1\|$, and therefore

$$|\det(A)| = \|\vec{v}_1\| \|\vec{v}_2 \perp\|.$$
The Determinant as Area, Volume, and Generalized Volume

We can use Gram-Schmidt (QR-factorization) to generalize to the $n \times n$ case:

- Given an invertible $n \times n$ matrix $A = [\vec{v}_1 \ \cdots \ \vec{v}_n]$, we can write $A = QR$ where $Q$ is an orthogonal matrix, and $R$ is an upper triangular matrix:

$$|\det(A)| = |\det(Q)| \cdot |\det(R)|$$

where $|\det(Q)| = 1$ since it is orthogonal.

- The diagonal entries of $R$ are given by

$$r_{11} = \|\vec{v}_1\|, \quad r_{jj} = \|\vec{v}_j^\perp\|, \quad j \geq 2$$

where $\|\vec{v}_j^\perp\|$ is the component of $\vec{v}_j$ perpendicular to $\operatorname{span}(\vec{v}_1, \ldots, \vec{v}_{j-1})$. 

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Determinant: Geometrical View, Cramer’s Rule — (8/22)
The Determinant as Area, Volume, and Generalized Volume

Theorem (The Determinant in Terms of the Columns)

*If* \( A \) *is an* \( n \times n \) *with columns* \( \vec{v}_1, \ldots, \vec{v}_n \), *then*

\[
|\det(A)| = \|\vec{v}_1\| \|\vec{v}_2\| \cdots \|\vec{v}_n\|, 
\]

*where* \( \|\vec{v}_j\| \) *is the component of* \( \vec{v}_j \) *perpendicular to* \( \text{span}(\vec{v}_1, \ldots, \vec{v}_{j-1}) \).

*(Notation from the Gram-Schmidt process).*

Gram-Schmidt \( QR \)-factorization may seem like a painful thing (it requires about 2–3 times the work of just computing \( \text{rref}(A) \)), but it gives us a useful results:

1. \( Q \) contains an orthonormal basis for \( \text{im}(A) \)
2. \( R \) contains the transformation from (non-orthogonal) \( A \)-coordinates to (orthonormal) \( Q \)-coordinates.
3. \( R \) can be used to compute \( \det(A) \) using only \( (n - 1) \) multiplications.
The Determinant as Area, Volume, and Generalized Volume

**Figure:** With \((\vec{v}_1, \vec{v}_2, \vec{v}_3) = (\vec{b}, \vec{c}, \vec{a})\), we have

\[
\text{Volume} = \left( \|\vec{v}_1\| \|\vec{v}_2^\perp\| \|\vec{v}_3^\perp\| \right)
\]

Base Area Height

Theorem (Volume of a Parallelepiped in \(\mathbb{R}^3\))

Consider a \(3 \times 3\) matrix

\[
A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{bmatrix}
\]

Then the volume of the parallelepiped defined by \(\vec{v}_1, \vec{v}_2, \text{ and } \vec{v}_3\) is \(|\det(A)|\).
The Determinant as Area, Volume, and Generalized Volume

Definition (Parallelepipeds in $\mathbb{R}^n$)
Consider the vectors $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$. The $m$-Parallelepiped defined by these vectors is:

$$\{ \vec{y} \in \mathbb{R}^n : \vec{y} = c_1 \vec{v}_1 + \cdots + c_m \vec{v}_m, \ c_k \in [0, 1] \}$$

The $m$-volume $V(\vec{v}_1, \ldots, \vec{v}_m)$ of this $m$-Parallelepiped is defined recursively by: $V(\vec{v}_1) = \| \vec{v}_1 \|$, and

$$V(\vec{v}_1, \ldots, \vec{v}_m) = V(\vec{v}_1, \ldots, \vec{v}_{m-1}) \| \vec{v}_m^\perp \|,$$

or equivalently

$$V(\vec{v}_1, \ldots, \vec{v}_m) = \| \vec{v}_1 \| \| \vec{v}_2^\perp \| \cdots \| \vec{v}_m^\perp \|.$$
We can leverage the $QR$-factorization:

Let $A \in \mathbb{R}^{n \times m}$, with columns $\vec{v}_1, \ldots, \vec{v}_m$. If the columns are linearly independent, consider the $QR$-factorization $A = QR$. Then

$$A^T A = (QR)^T (QR) = (R^T Q^T) (QR) = R^T (Q^T Q) R = R^T R,$$

so that

$$\det(A^T A) = \det(R^T R) = (\det(R))^2 = (r_{11} r_{22} \cdots r_{mm})^2 = (\|\vec{v}_1\| \|\vec{v}_2\| \cdots \|\vec{v}_m\|)^2 = (V(\vec{v}_1, \ldots, \vec{v}_m))^2$$
The Determinant as Area, Volume, and Generalized Volume

Theorem (Volume of Parallelepipeds in $\mathbb{R}^n$)

Consider the vectors $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$. Then the $m$-Volume of the $m$-Parallelepiped defined by these vectors is:

$$\sqrt{\det(A^TA)},$$

where $A$ is the $n \times m$ matrix with columns $\vec{v}_1, \ldots, \vec{v}_m$.

In particular, when $m = n$, this volume is

$$|\det(A)|$$
Consider a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. We have discussed how such a transform impacts lengths, and angles. For a transform $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ it also makes sense to think about the 2-volume (aka “The Area”); and for $\mathbb{R}^n \rightarrow \mathbb{R}^n$ we can discuss the $m$-Volume(s).
We start in the \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) case, and let the “input area” (\( \Omega \)) be the unit square (with area 1), described by the two vectors \( \vec{e}_1 \), and \( \vec{e}_2 \).

The “output area” (\( T(\Omega) \)), is then described by \( A\vec{e}_1 = \vec{v}_1 \), and \( A\vec{e}_2 = \vec{v}_2 \), i.e. the parallelepiped spanned by the columns of \( A \); here the area is \(|\det(A)|\).

The expansion factor is

\[
\frac{\text{area of } T(\Omega)}{\text{area of } \Omega} = \frac{|\det(A)|}{1} = |\det(A)|.
\]
The Determinant as Expansion Factor

If the input parallelepiped is described by two vectors \( \vec{w}_1 \), and \( \vec{w}_2 \), then the original area is \( |\det(B)| \), where \( B = [\vec{w}_1 \quad \vec{w}_2] \).

The “output area” \( (T(\Omega)) \), is then described by \( A\vec{w}_1 = \vec{v}_1 \), and \( A\vec{w}_2 = \vec{v}_2 \); so the area of \( T(\Omega) \) is given by

\[
|\det([A\vec{w}_1 \quad A\vec{w}_2])| = |\det(AB)| = |\det(A)| \cdot |\det(B)|.
\]

The expansion factor is

\[
\frac{\text{area of } T(\Omega)}{\text{area of } \Omega} = \frac{|\det(A)| \cdot |\det(B)|}{|\det(B)|} = |\det(A)|.
\]
The Determinant as Expansion Factor

Theorem (Expansion Factor)

Consider a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $T(\vec{x}) = A\vec{x}$. Then $|\det(A)|$ is the expansion factor

\[
\frac{\text{area of } T(\Omega)}{\text{area of } \Omega}
\]

of $T$ on parallelograms $\Omega$.

Likewise, for linear transformations $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defined by $T(\vec{x}) = A\vec{x}$, $|\det(A)|$ is the expansion factor of $T$ on $n$-parallelepipeds:

\[
V(A\vec{v}_1, \ldots, A\vec{v}_n) = |\det(A)| \cdot V(\vec{v}_1, \ldots, \vec{v}_n),
\]

for all vectors $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n$. 
The Determinant as Expansion Factor

Stating the “Obvious”

Since $\det(AB) = \det(A) \det(B)$, which of course holds for $B = A^{-1}$, the expansion factors satisfy:

$$|\det(AB)| = |\det(A)| |\det(B)|,$$

and

$$|\det(A^{-1})| = \frac{1}{|\det(A)|}.$$

Stating the “Less Obvious”

It is possible to show that the expansion factor associated with a linear transformation $T(\vec{x}) = A\vec{x}$ (as we have defined it) holds for any region $\Omega$ (not just parallelograms or $n$-parallelepipeds).
Theorem (Cramer’s Rule)

Consider the linear system

\[ A\vec{x} = \vec{b} \]

where \( A \) is an invertible \( n \times n \) system. The components \( x_i \) of the solution vector \( \vec{x} \) are

\[ x_i = \frac{\det(A_{\vec{b} - i})}{\det(A)}, \]

where \( A_{\vec{b} - i} \) is the matrix obtained by replacing the \( i^{th} \) column of \( A \) by \( \vec{b} \).

Peter’s Postulate

Solving linear systems using Cramer’s Rule is a BAD IDEA. — We need to compute \((n + 1)\) determinants of size \((n \times n)\).
More “Great” Ideas...

Recall:

Definition (Minors)
For an \( n \times n \) matrix \( A \), let \( A_{ij} \) be the matrix obtained by omitting the \( i^{th} \) row, and \( j^{th} \) column of \( A \). The determinant of the \((n - 1) \times (n - 1)\) matrix \( A_{ij} \) is called a minor of \( A \).

Now:

Definition (The Classical Adjoint)
The classical adjoint \( M = \text{adj}(A) \) of an invertible \( n \times n \) matrix, is the matrix whose \( ij^{th} \) entry \( m_{ij} = (-1)^{i+j} \det(A_{ij}) \). Yes, we have to compute \( n^2 \) \((n - 1) \times (n - 1)\) determinants to build the adjoint! With “only” one more \( n \times n \) determinant, we can express the inverse:

\[
A^{-1} = \frac{1}{\det(A)} \text{adj}(A).
\]
Suggested Problems 6.3

Available on Learning Glass videos:
6.3 — 1, 3, 5, 7, 9, 11, 13, 19-20-21
Lecture – Book Roadmap

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<td>§5.1, §5.2, §5.3</td>
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<tr>
<td>6.3</td>
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* Strang does not talk about the combinatorial (pattern) definition of the determinant.