Student Learning Objectives

SLOs: Determinant: Geometrical View, Cramer’s Rule

The Determinant: Redux

Geometrical Interpretations (“Volume”)
The Determinant as Expansion Factor
Cramer’s Rule: The Worst Idea In Linear Algebra(?)

Suggested Problems

Suggested Problems 6.3
Lecture–Book Roadmap

Supplemental Material

Metacognitive Reflection
Problem Statements 6.3
Application: Ordinary Differential Equations (ODEs)
After this lecture you should:

- Know that $|\det(A)| = 1 \iff$ Orthogonal Matrix; e.g. rotation or reflection.
- Be familiar with the Interpretation of the determinant as an $m$-Volume; and/or an expansion factor.
- Forget about Cramer’s Rule: Don’t Use It!
We have several methods for computing the determinant:

- **The “pattern method”** (straight from definition),
  - Best when the matrix has lots of zeros.

- **Laplace co-factor expansion** (streamlined “pattern method”),
  - This will be our method of choice when we find eigenvalues (of home-work sized matrices).

- **Row-reduction method**;
  - Best “general purpose” direct computational method.

next we ponder its geometrical interpretations.
Orthogonal Matrices

Example (Determinant of an Orthogonal Matrix)

What are the possible values of det(A), when A is orthogonal?

**Answer:** Orthogonal means that $A^T A = I_n$, so we have

$$1 = \det(I_n) = \det(A^T A) = \det(A^T)\det(A) = (\det(A))^2,$$

therefore $\det(A) = \pm 1$.

Theorem (The Determinant of an Orthogonal Matrix)

*The determinant of an orthogonal matrix is either 1 or $-1$.**
Orthogonal Matrices

Example (The Determinant of a Rotation Matrix)

Let

$$M(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Then

$$\det(M(\theta)) = (\cos(\theta))^2 + (\sin(\theta))^2 = 1.$$ 

Definition (Rotation Matrix)

An orthogonal $n \times n$ matrix $A$, with $\det(A) = 1$ is called a rotation matrix, and the linear transformation $T(\vec{x}) = A\vec{x}$ is called a rotation.
Example (The Determinant of a Reflection Matrix)

Let

\[ M(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix} \]

Then

\[ \det(M(\theta)) = -(\cos(\theta))^2 - (\sin(\theta))^2 = -1. \]

**Figure:** Reflections with \( M(4\pi/3) \), and \( M(7\pi/4) \).
We have given the $2 \times 2$ determinant a geometrical interpretation, with

$$\det(A) = \det \left( \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \right) = \sin(\theta) \|\vec{v}_1\| \|\vec{v}_2\|,$$

$|\det(A)|$ is the area of the parallelogram spanned by the vectors $\vec{v}_1$, and $\vec{v}_2$. $|\sin(\theta)| \|\vec{v}_2\| = \|\vec{v}_2 \perp \vec{v}_1\|$, and therefore

$$|\det(A)| = \|\vec{v}_1\| \|\vec{v}_2 \perp\|.$$
We use Gram-Schmidt ($QR$-factorization) to generalize to the $n \times n$ case:

- Given an invertible $n \times n$ matrix $A = [\vec{v}_1 \cdots \vec{v}_n]$, we can find $A = QR$ where $Q$ is an orthogonal matrix, and $R$ is an upper triangular matrix:

$$|\det(A)| = |\det(Q)| \cdot |\det(R)|$$

where $|\det(Q)| = 1$ since it is orthogonal.

- The diagonal entries of $R$ are given by

$$r_{11} = \|\vec{v}_1\|, \quad r_{jj} = \|\vec{v}_j^\perp\|, \quad j \geq 2$$

where $\|\vec{v}_j^\perp\|$ is the component of $\vec{v}_j$ perpendicular to $\text{span}(\vec{v}_1, \ldots, \vec{v}_{j-1})$. 
The Determinant as Area, Volume, and Generalized Volume

Theorem (The Determinant in Terms of the Columns)

If $A$ is an $n \times n$ with columns $\vec{v}_1, \ldots, \vec{v}_n$, then

$$|\det(A)| = \|\vec{v}_1\| \|\vec{v}_2^\perp\| \ldots \|\vec{v}_n^\perp\|,$$

where $\|\vec{v}_j^\perp\|$ is the component of $\vec{v}_j$ perpendicular to $\text{span}(\vec{v}_1, \ldots, \vec{v}_{j-1})$.

(Notation from the Gram-Schmidt process).

Gram-Schmidt QR-factorization may seem like a painful thing (it requires about 2–3 times the work of just computing $\text{rref}(A)$, but it gives us a useful results:

1. $Q$ contains an orthonormal basis for $\text{im}(A)$
2. $R$ contains the transformation from (non-orthogonal) $A$-coordinates to (orthonormal) $Q$-coordinates.
3. $R$ can be used to compute $\det(A)$ using only $(n - 1)$ multiplications.

Peter Blomgren, ⟨blomgren.peter@gmail.com⟩ Determinant: Geometrical View, Cramer’s Rule — (10/38)
The Determinant as Area, Volume, and Generalized Volume

Figure: With \((\vec{v}_1, \vec{v}_2, \vec{v}_3) = (\vec{b}, \vec{c}, \vec{a})\), we have

\[
\text{Volume} = \left| \begin{vmatrix} \|\vec{v}_1\| & \|\vec{v}_2\| & \|\vec{v}_3\| \\ \|\vec{v}_1\| & \|\vec{v}_2\| & \|\vec{v}_3\| \end{vmatrix} \right|
\]

Theorem (Volume of a Parallelepiped in \(\mathbb{R}^3\))

Consider a 3 \(\times\) 3 matrix \(A = [\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_3]\). Then the volume of the parallelepiped defined by \(\vec{v}_1, \vec{v}_2, \text{ and } \vec{v}_3\) is \(|\det(A)|\).
The Determinant as Area, Volume, and Generalized Volume

Definition (Parallelepipeds in $\mathbb{R}^n$)

Consider the vectors $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$. The $m$-Parallelepiped defined by these vectors is:

\[
\{ \vec{y} \in \mathbb{R}^n : \vec{y} = c_1 \vec{v}_1 + \cdots + c_m \vec{v}_m, \ c_k \in [0, 1] \} 
\]

The $m$-volume $V(\vec{v}_1, \ldots, \vec{v}_m)$ of this $m$-Parallelepiped is defined recursively by: $V(\vec{v}_1) = \|\vec{v}_1\|$, and

\[
V(\vec{v}_1, \ldots, \vec{v}_m) = V(\vec{v}_1, \ldots, \vec{v}_{m-1}) \|\vec{v}_m^\perp\|, \\
\]

or equivalently

\[
V(\vec{v}_1, \ldots, \vec{v}_m) = \|\vec{v}_1\| \|\vec{v}_2^\perp\| \cdots \|\vec{v}_m^\perp\|. 
\]
We can leverage the QR-factorization:

Let $A \in \mathbb{R}^{n \times m}$, with columns $\vec{v}_1, \ldots, \vec{v}_m$. If the columns are linearly independent, consider the QR-factorization $A = QR$. ($Q \in \mathbb{R}^{n \times n}$, $R \in \mathbb{R}^{n \times m}$) Then

$$A^T A = (QR)^T (QR) = (R^T Q^T)(QR) = R^T (Q^T Q)R = R^T R,$$

so that

$$\det(A^T A) = \det(R^T R) = (\det(R))^2 = (r_{11}r_{22} \cdots r_{mm})^2$$

$$= (\|\vec{v}_1\| \|\vec{v}_2^\perp\| \cdots \|\vec{v}_m^\perp\|)^2 = (V(\vec{v}_1, \ldots, \vec{v}_m))^2$$
Consider the vectors $\vec{v}_1, \ldots, \vec{v}_m \in \mathbb{R}^n$. Then the $m$-Volume of the $m$-Parallelepiped defined by these vectors is:

$$\sqrt{\det(A^T A)},$$

where $A$ is the $n \times m$ matrix with columns $\vec{v}_1, \ldots, \vec{v}_m$.

In particular, when $m = n$, this volume is

$$|\det(A)|$$
Consider a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$. We have discussed how such a transform impacts lengths, and angles. For a transform $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ it also makes sense to think about the 2-volume (aka “The Area”); and for $\mathbb{R}^n \rightarrow \mathbb{R}^n$ we can discuss the $m$-Volume(s).
The Determinant as Expansion Factor

We start in the $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ case, and let the “input area” ($\Omega$) be the unit square (with area 1), described by the two vectors $\vec{e}_1$, and $\vec{e}_2$.

The “output area” ($T(\Omega)$), is then described by $A\vec{e}_1 = \vec{v}_1$, and $A\vec{e}_2 = \vec{v}_2$, i.e. the parallelepiped spanned by the columns of $A$; here the area is $|\det(A)|$.

The expansion factor is

\[
\frac{\text{area of } T(\Omega)}{\text{area of } \Omega} = \frac{|\det(A)|}{1} = |\det(A)|.
\]
The Determinant as Expansion Factor

If the input parallelepiped is described by two vectors \( \vec{w}_1 \), and \( \vec{w}_2 \), then the original area is \(|\det(B)|\), where \( B = \begin{bmatrix} \vec{w}_1 & \vec{w}_2 \end{bmatrix} \).

The “output area” \((T(\Omega))\), is then described by \( A\vec{w}_1 = \vec{v}_1 \), and \( A\vec{w}_2 = \vec{v}_2 \); so the area of \( T(\Omega) \) is given by

\[
|\det(\begin{bmatrix} A\vec{w}_1 & A\vec{w}_2 \end{bmatrix})| = |\det(AB)| = |\det(A)| |\det(B)|.
\]

The expansion factor is

\[
\frac{\text{area of } T(\Omega)}{\text{area of } \Omega} = \frac{|\det(A)| |\det(B)|}{|\det(B)|} = |\det(A)|.
\]
The Determinant as Expansion Factor

**Theorem (Expansion Factor)**

Consider a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\vec{x}) = A\vec{x}$. Then $|\det(A)|$ is the expansion factor of $T$ on parallelograms $\Omega$.

Likewise, for linear transformations $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(\vec{x}) = A\vec{x}$, $|\det(A)|$ is the expansion factor of $T$ on $n$-parallelepipeds:

$$V(A\vec{v}_1, \ldots, A\vec{v}_n) = |\det(A)| \cdot V(\vec{v}_1, \ldots, \vec{v}_n),$$

for all vectors $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n$. 

Peter Blomgren, ⟨blomgren.peter@gmail.com⟩

Determinant: Geometrical View, Cramer’s Rule — (18/38)
The Determinant as Expansion Factor

Stating the "Obvious"

Since \( \det(AB) = \det(A) \det(B) \), which of course holds for \( B = A^{-1} \), the expansion factors satisfy:

\[
|\det(AB)| = |\det(A)| |\det(B)|,
\]

and

\[
|\det(A^{-1})| = \frac{1}{|\det(A)|}.
\]

Stating the "Less Obvious"

It is possible to show that the expansion factor associated with a linear transformation \( T(\vec{x}) = A\vec{x} \) (as we have defined it) holds for any region \( \Omega \) (not just parallelograms or \( n \)-parallelepipeds).
Cramer’s Rule

Theorem (Cramer’s Rule)

Consider the linear system

\[ A\vec{x} = \vec{b} \]

where \( A \) is an invertible \( n \times n \) system. The components \( x_i \) of the solution vector \( \vec{x} \) are

\[ x_i = \frac{\det(A_{\vec{b},i})}{\det(A)}, \]

where \( A_{\vec{b},i} \) is the matrix obtained by replacing the \( i^{th} \) column of \( A \) by \( \vec{b} \).

Peter’s Postulate

Solving linear systems using Cramer’s Rule is a BAD IDEA. — We need to compute \((n + 1)\) determinants of size \((n \times n)\).
More “Great” Ideas...

Recall:

**Definition (Minors)**

For an $n \times n$ matrix $A$, let $A_{ij}$ be the matrix obtained by omitting the $i^{th}$ row, and $j^{th}$ column of $A$. The determinant of the $(n - 1) \times (n - 1)$ matrix $A_{ij}$ is called a *minor* of $A$.

**Now:**

**Definition (The Classical Adjoint)**

The classical adjoint $M = \text{adj}(A)$ of an invertible $n \times n$ matrix, is the matrix whose $ij^{th}$ entry $m_{ij} = (-1)^{i+j} \det(A_{ij})$. Yes, we have to compute $n^2$ $(n - 1) \times (n - 1)$ determinants to build the adjoint! With “only” one more $n \times n$ determinant, we can express the inverse:

$$A^{-1} = \frac{1}{\det(A)} \text{adj}(A).$$
Available on Learning Glass videos:
6.3 — 1, 3, 5, 7, 9, 11, 13, 19-20-21
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* Strang does not talk about the combinatorial (pattern) definition of the determinant.
### Metacognitive Exercise — Thinking About Thinking & Learning

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Peter Blomgren, ⟨blomgren.peter@gmail.com⟩

Determinant: Geometrical View, Cramer’s Rule — (24/38)
(6.3.1) Find the area of the parallelogram defined by
\[
\begin{bmatrix}
3 \\
7
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
8 \\
2
\end{bmatrix}.
\]

(6.3.3) Find the area of the triangle with corners in
\[
\begin{bmatrix}
5 \\
7
\end{bmatrix}, \quad
\begin{bmatrix}
4 \\
3
\end{bmatrix}, \quad \text{and} \quad
\begin{bmatrix}
10 \\
1
\end{bmatrix}.
\]
The tetrahedron defined by three vectors $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3 \in \mathbb{R}^3$ is the set of all vectors of the form $c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3$, where $c_i \geq 0$, and $c_1 + c_2 + c_3 \leq 1$. Explain why the volume is one sixth of the volume of the parallelepiped defined by $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$. 
(6.3.7) Find the area of the region with corners in \([-7, 7], [-5, -6], [-4, -3], \) and \([5, 5] \).

(6.3.9) If \(\vec{v}_1\) and \(\vec{v}_2\) are linearly independent vectors in \(\mathbb{R}^2\), what is the relationship between \(\det([\vec{v}_1 \quad \vec{v}_2])\) and \(\det([\vec{v}_1 \quad \vec{v}_2^\perp])\), where \(\vec{v}_2^\perp\) is the component of \(\vec{v}_2\) orthogonal to \(\vec{v}_1\).

(6.3.11) Consider a linear transformation \(T(\vec{x}) = A\vec{x}\) from \(\mathbb{R}^2\) to \(\mathbb{R}^2\). Suppose for two vectors \(\vec{v}_1\) and \(\vec{v}_2\) in \(\mathbb{R}^2\) we have \(T(\vec{v}_1) = 3\vec{v}_1\) and \(T(\vec{v}_2) = 4\vec{v}_2\). What can you say about \(\det(A)\)? Explain in detail.
(6.3.13) Find the 2-volume (aka “area”) of the 2-parallelepiped (parallelogram) defined by the two vectors

\[
\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \text{ and } \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.
\]

(6.3.19) A basis \((\vec{v}_1, \vec{v}_2, \vec{v}_3)\) of \(\mathbb{R}^3\) is called positively oriented if \(\vec{v}_1\) encloses an acute angle with \(\vec{v}_2 \times \vec{v}_3\). Illustrate with a sketch. Show that the basis is positively oriented if-and-only-if \(\det([\vec{v}_1 \, \vec{v}_2 \, \vec{v}_3])\) is positive.

(6.3.20) We say that a linear transformation \(T : \mathbb{R}^3 \to \mathbb{R}^3\) preserves orientation if it transforms any positively oriented basis into another positively oriented basis. Explain why a linear transformation \(T(\vec{x}) = A\vec{x}\) preserves orientation if-and-only-if \(\det(A) > 0\).
Arguing geometrically, determine whether the following orthogonal transformations from \( \mathbb{R}^3 \) to \( \mathbb{R}^3 \) preserve, or reverse orientation:

a. Reflection about a plane.
b. Reflection about a line.
c. Reflection about the origin.
Consider the nonhomogeneous problem

$$y'' + p(t)y' + q(t)y = g(t),$$

where $p(t)$, $q(t)$, and $g(t)$ are given continuous functions.

Assume we know the **homogeneous solution** (for the case $g(t) = 0$):

$$y_h(t) = c_1y_1(t) + c_2y_2(t)$$

We try a **general solution** of the form $y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$, where the functions $u_1(t)$ and $u_2(t)$ are to be determined.

Differentiating gives:

$$y'(t) = u_1(t)y_1'(t) + u_2(t)y_2'(t) + u_1'(t)y_1(t) + u_2'(t)y_2(t)$$
We relate $u_1(t)$ and $u_2(t)$ (one degree of freedom), be setting

$$u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0,$$

this simplifies the derivative of the general solution.

We use the fact that $y_1(t)$ and $y_2(t)$ satisfy the homogeneous equation... 

$\text{“}(t)\text{”})$:

$$y'' = u_1y_1'' + u_2y_2'' + u_1'y_1 + u_2'y_2$$

$$y'' + py' + qy = [u_1y_1'' + u_2y_2'' + u_1'y_1 + u_2'y_2] + p[u_1y_1' + u_2y_2'] + q[u_1y_1 + u_2y_2] = g$$

$$y'' + py' + qy = [u_1'y_1 + u_2'y_2] + u_1[y'' + py' + qy_1] + u_2[y'' + py' + qy_2] = g$$

... and we are left with

$$u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t)$$
This gives two **linear algebraic equations** in $u_1'$ and $u_2'$

\[
u_1'(t)y_1(t) + u_2'(t)y_2(t) = 0
\]

\[
u_1'(t)y_1'(t) + u_2'(t)y_2'(t) = g(t),
\]

or in matrix form

\[
\begin{bmatrix}
y_1(t) & y_2(t) \\
y_1'(t) & y_2'(t)
\end{bmatrix}
\begin{bmatrix}
u_1'(t) \\
u_2'(t)
\end{bmatrix}
= \begin{bmatrix} 0 \\
g(t)
\end{bmatrix}
\]

\[
\underbrace{A(t)}
\]

As long as the columns of $A(t)$ are linearly independent, we get solutions

[NOTES 2.4]:

\[
\begin{bmatrix}
u_1'(t) \\
u_2'(t)
\end{bmatrix}
= \frac{1}{\det(A(t))}
\begin{bmatrix}
y_2'(t) & -y_2(t) \\
-y_1'(t) & y_1(t)
\end{bmatrix}
\begin{bmatrix} 0 \\
g(t)
\end{bmatrix}
= \frac{1}{\det(A(t))}
\begin{bmatrix}
-y_2(t) & g(t) \\
y_1(t) & g(t)
\end{bmatrix}
\]
In the “ODE Universe,” \( \det(A(t)) = y_1(t)y'_2(t) - y'_1(t)y_2(t) \) is usually referred to as the “Wronskian”, and sometimes denoted \( W[y_1, y_2](t) \).

We can now integrate

\[
  u_1(t) = -\int_t^\tau \frac{y_2(\tau)g(\tau)}{\det(A(\tau))} \, d\tau + C_1, \quad u_2(t) = \int_t^\tau \frac{y_1(\tau)g(\tau)}{\det(A(\tau))} \, d\tau + C_2,
\]

and the general solution is given by

\[
  y(t) = u_1(t)y_1(t) + u_2(t)y_2(t);
\]

where \( u_1(t) \) and \( u_2(t) \) be given explicitly, or in integral form.
Theorem (Variation of Parameters)

Consider the nonhomogeneous equation

\[ y'' + p(t)y' + q(t)y = g(t), \]

If the functions \( p(t) \), \( q(t) \), and \( g(t) \) are continuous on an open interval \( I \subset \mathbb{R}^n \), and if \( y_1 \) and \( y_2 \) form a fundamental set of solutions of the homogeneous equation. Then a particular solution of the nonhomogeneous problem is

\[ y_p(t) = -y_1(t) \int_{t_0}^{t} \frac{y_2(s)g(s)}{W[y_1, y_2](s)} \, ds + y_2(t) \int_{t_0}^{t} \frac{y_1(s)g(s)}{W[y_1, y_2](s)} \, ds, \]

where \( t_0 \in I \). The general solution is

\[ y(t) = c_1y_1(t) + c_2y_2(t) + y_p(t). \]
We consider the nonhomogeneous problem

\[ y'' + 4y = 3 \sin(t) \]

The homogeneous problem

\[ y'' + 4y = 0 \]

has two **linearly independent solutions**

\[ y_1(t) = \cos(2t), \quad \text{and} \quad y_2(t) = \sin(2t). \]

**Variation of Parameters** suggests

\[ y(t) = u_1(t) \cos(2t) + u_2(t) \sin(2t), \]

where the functions \( u_1(t) \) and \( u_2(t) \) are to be determined.
Using the “recipe” we get

\[
\begin{bmatrix}
   y_1(t) & y_2(t) \\
   y'_1(t) & y'_2(t)
\end{bmatrix}
\begin{bmatrix}
   u'_1(t) \\
   u'_2(t)
\end{bmatrix}
= \begin{bmatrix}
   \cos(2t) & \sin(2t) \\
   -2\sin(2t) & 2\cos(2t)
\end{bmatrix}
\begin{bmatrix}
   u'_1(t) \\
   u'_2(t)
\end{bmatrix}
= \begin{bmatrix}
   g(t) \\
   0
\end{bmatrix}
\]

The **Wronskian** satisfies:

\[
W[\cos(2t), \sin(2t)](t) = \det(A(t)) = 2\cos^2(2t) + 2\sin^2(2t) \equiv 2
\]

So that

\[
u'_1(t) = \frac{-3\sin(t)\sin(2t)}{2}, \quad u'_2(t) = \frac{3\sin(t)\cos(2t)}{2}
\]

The rest is “just” trig-identities and integration...
Things we have forgotten:

\[
\sin(2t) = 2 \cos(t) \sin(t), \quad \cos(2t) = 2 \cos^2(t) - 1
\]

\[
u_1'(t) = \frac{-3 \sin(t) \sin(2t)}{2} = -3 \cos(t) \sin^2(t)
\]

\[
u_2'(t) = \frac{3 \sin(t) \cos(2t)}{2} = 3 \cos^2(t) \sin(t) - \frac{3}{2} \sin(t)
\]

Which gives us

\[
u_1(t) = -\sin^3(t) + C_1, \quad \nu_2(t) = \frac{3}{2} \cos(t) - \cos^3(t) + C_2
\]
The General Solution is

\[ y(t) = u_1(t) \cos(2t) + u_2(t) \sin(2t) \]
\[ = (-\sin^3(t) + C_1) \cos(2t) + \left(\frac{3}{2} \cos(t) - \cos^3(t) + C_2\right) \sin(2t) \]
\[ = -\sin^3(t) \cos(2t) + \frac{3}{2} \cos(t) \sin(2t) - \cos^3(t) \sin(2t) \]
\[ + C_1 \cos(2t) + C_2 \sin(2t) \]
\[ = \sin(t) + C_1 \cos(2t) + C_2 \sin(2t) \]

Somebody tried to convince me that Cramer’s rule was a nice way to solve the linear 2×2 linear system... but, no, I still prefer the route taken in these notes.

**Disclaimer:** I’m sure I “lost” a minus-sign somewhere...